Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 7

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Exercise 1. *Transformation behavior of bilinear quantities involving γ-matrices*
Let $\psi : \mathbb{M} \to \mathbb{C}^4$. Using the transformation behavior under proper Lorentz transformations $\Lambda \in \mathcal{L}_+^\uparrow$, $x \to x' = \Lambda x$,

$$\psi'(x') = S[\Lambda]\psi(x) \tag{1}$$

with invertible 4×4 matrices $S[\Lambda]$ such that

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu \gamma^\nu, \quad \mu = 0,1,2,3, \tag{2}$$

and $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$ prove that the quantities

$$S(x) = \overline{\psi}(x)\psi(x), \quad V^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x), \quad T^{\mu\nu}(x) = \overline{\psi}(x)\gamma^\mu\gamma^\nu\psi(x) \tag{3}$$

transform as scalar, vector and tensor fields, respectively.

Exercise 2. *Solution of the massless Dirac equation in 1+1 dimensions.*
Let $\psi : \mathbb{R}^2 \to \mathbb{C}^2$. Choose the representation

$$\gamma^0 = \sigma^1, \quad \gamma^1 = \sigma^1\sigma^3 \tag{4}$$

for the 1+1-dimensional Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad \text{(sum over } \mu = 0,1). \tag{5}$$

In the case $m = 0$, find the solution satisfying the initial condition

$$\psi(0,\cdot) = f(\cdot) \in C^1(\mathbb{R}, \mathbb{C}^2). \tag{6}$$

Hint. In the mass-less case and for $m = 0$, write out the equation in components. You should see that it is simple enough so that you can find its general solution. Compare this general solution with the initial data.

Exercise 3. *Boundary conditions and probability conservation.*
Consider the Dirac equation on the half-space $\mathbb{H} = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 : x^1 > 0\}$.

(a) Identify possible linear boundary conditions for ψ which ensure that there is no probability flux through the boundary $\partial\mathbb{H}$.
Hints: These boundary conditions should involve only two independent linear relations between the components of ψ. The probability flux density through a hypersurface $\Sigma \subset \mathbb{R}^4$ is defined as $j^\mu(x)n_\mu(x)$ where n is the normal covector field at Σ and $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$.

(b) Choose a boundary condition from (a). Show that for a solution ψ of the Dirac equation which satisfies this boundary condition, the total probability integral $P(\Sigma \cap \mathbb{H}) = \int_{\Sigma\cap\mathbb{H}} d\sigma(x)\, n_\mu(x)\, \overline{\psi}(x)\gamma^\mu\psi(x)$ is independent of the space-like hyperplane Σ. What does this mean for the solution theory?