Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 8

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Exercise 1. *Discrete transformations of the Dirac equation.*
Let $\psi \in C^1(\mathbb{R}^4, \mathbb{C}^4)$ be a solution of the Dirac equation.

(a) Show that for time reversal, $x = (t, \mathbf{x}) \to x' = (-t, \mathbf{x})$, $\psi'(x') = B\psi^*(x)$ with $B(\gamma^{0*}, -\boldsymbol{\gamma}^*)B^{-1} = (\gamma^0, \boldsymbol{\gamma})$, $B = \gamma^1\gamma^2$ satisfies the Dirac equation in x'.

(b) Prove that the Dirac current transforms as follows under parity and time reversal:

$$\overline{\psi}'(x')\gamma^\mu\psi'(x') = (-1)^{1-\delta_0^\mu}\,\overline{\psi}(x)\gamma^\mu\psi(x). \tag{1}$$

Parity is here defined by $x = (t, \mathbf{x}) \to x' = (t, -\mathbf{x})$, $\psi'(x') = \gamma^0\psi(x)$.

(c) Show that if $\psi(x)$ satisfies the equation

$$[\gamma^\mu(i\partial_\mu - eA_\mu(x)) - m]\psi(x) = 0. \tag{2}$$

with a vector field $A : \mathbb{R}^4 \to \mathbb{R}^4$, then the charge conjugated wave function $\psi'(x) = C\overline{\psi}^T(x)$ with $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$, $\mu = 0, 1, 2, 3$, $C = i\gamma^0\gamma^2$ satisfies (2) with charge $+e$ instead of $-e$.

Exercise 2. *Total probability conservation in all Lorentz frames.*
Let $\Lambda \in \mathcal{L}_+^\uparrow$ be a Lorentz boost and let $(t', \mathbf{x}') = \Lambda(t, \mathbf{x})$. Moreover, let $\psi \in C^1(\mathbb{R}^4, \mathbb{C}^4)$ be a solution of the Dirac equation which lies in $L^2(\Sigma, \mathbb{C}^4)$ for every space-like hyperplane $\Sigma \subset \mathbb{M}$. Prove in detail that then the following equality holds (probability conservation in the two inertial frames related by Λ):

$$\int d^3\mathbf{x}\,\psi^\dagger(t, \mathbf{x})\psi(t, \mathbf{x}) = \int d^3\mathbf{x}'\,\psi'^\dagger(t', \mathbf{x}')\psi'(t', \mathbf{x}') \quad \forall t, t' \in \mathbb{R}. \tag{3}$$

Exercise 3. *Solution of the Dirac equation in 1+1 dimensions on a half space.*
Consider the massless Dirac equation in 1+1 dimensions on $\mathbb{H} := \{(t, x) \in \mathbb{R}^2 : x > 0\}$,

$$i\gamma^\mu\partial_\mu\psi(t, x) = 0, \quad x > 0, t \in \mathbb{R}, \tag{4}$$

where $\psi = (\psi_1, \psi_2)$, $\gamma^0 = \sigma^1$ and $\gamma^1 = \sigma^1\sigma^3$ and $\mu = 0, 1$.

(a) Let n be the exterior unit normal vector field at $\partial\mathbb{H}$. Find the most general linear (but time-independent) boundary condition for ψ at $x^1 = 0$ which ensures that the probability flux $n_\mu\overline{\psi}\gamma^\mu\psi$ into $\partial\mathbb{H}$ vanishes. (The result should be one linear condition between the two components of ψ.)

(b) Solve the initial boundary value problem consisting of the Dirac equation, the boundary condition from (a) (or any other linear condition between the two components of ψ if you did not solve (a)) and the initial condition $\psi(0,x) = \psi_0(x),\ x \geq 0$ for some $\psi_0 \in \mathbb{C}^1([0,\infty),\mathbb{C}^2)$.

(c) Identify the conditions when the solution from (b) is (i) continuous and (ii) continuously differentiable across the lines $x = t$ and $x = -t$.

Hint. Find the general solution as on Sheet 7, Exercise 2. Consider carefully which part of the unknown functions in the general solution is determined by the initial condition. Use the boundary condition to determine the remaining part.