Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 9

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Exercise 1. *Negative energy solutions of the Dirac equation.*

(a) Find two linearly independent plane wave solutions of the Dirac equation of the form

$$\psi(x) = u_-^{(i)}(p) e^{-ip_\mu x^\mu}, \quad i = 1, 2, \tag{1}$$

where $u_-^{(i)}(p) \in \mathbb{C}^4,\ i = 1, 2$ and with energy $p^0 = -\sqrt{\mathbf{p}^2 + m^2}$.

(b) Show that the charge conjugation of $u_-^{(i)}$, $(u_-^{(i)})'(x) = C(\bar{u}_-^{(i)})^T(x)$, yields a positive-energy solution of the Dirac equation. How can one interpret this fact physically?

Exercise 2. *Dirac current for plane wave solutions.*

Calculate the probability current density $j^\mu(x)$ for:

(i) a positive energy plane wave solution of the Dirac equation (see formula in the lecture notes),

(ii) a negative energy plane wave solution.

Which qualitative behavior do you observe in these two cases?

Exercise 3. *Solution of the massive Dirac equation in 1+1 dimensions.*

Let $f \in C^2(\mathbb{R}, \mathbb{C}^2) \cap L^2(\mathbb{R}, \mathbb{C}^2)$ and pick a representation of the Clifford algebra relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,\mathbb{1}_2$ by 2×2 matrices $\gamma^\mu,\ \mu = 1, 2$ of your choice. Consider the following initial value problem for the Dirac equation in 1+1 dimensions with $m > 0$

$$\begin{cases} \psi(0, x) = f(x), & x \in \mathbb{R}, \\ (i\gamma^\mu\partial_\mu - m)\psi(t, x) = 0, & x \in \mathbb{R},\ t \in (0, \infty). \end{cases} \tag{2}$$

Now assume that this problem has a solution $\psi \in C^2(\mathbb{R}^2, \mathbb{C}^2)$. Prove that this solution is unique and give a formula which expresses the solution in terms of f.

Hint. Proceed as in the lecture for 1+3 dimensions: reduce the problem to an initial value problem for the KG equation in 1+1 dimensions with special initial data. For solving this new problem, the result of Exercise 2 on Sheet 4 will be helpful.