Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 10

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Exercise 1. *Dirac equation with external potential.*

(a) Let $\mathscr{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and let $\widehat{H}^{\text{Dirac}}$ be the Dirac Hamiltonian with domain $\mathscr{D}(\widehat{H}^{\text{Dirac}}) = H^1(\mathbb{R}^3)^4$. Given any bounded self-adjoint operator $\widehat{V} : \mathscr{H} \to \mathscr{H}$, prove that the initial value problem of the Dirac equation

$$\begin{cases} i\frac{d\psi(t)}{dt} = \left(\widehat{H}^{\text{Dirac}} + \widehat{V}\right)\psi(t), \\ \psi(0) = \psi_0 \in \mathscr{H} \end{cases} \tag{1}$$

has a unique Hilbert-space valued solution $\psi : \mathbb{R} \to \mathscr{H}$.

(b) Give at least one physically relevant example for a bounded self-adjoint operator $\widehat{V} : \mathscr{H} \to \mathscr{H}$.

Exercise 2. *Hilbert space associated with a space-like hyperplane.*

Let $\Sigma \subset \mathbb{M}$ be a space-like hyperplane. Show that the sesquilinear form

$$\langle \psi, \varphi \rangle_\Sigma = \int_\Sigma d\sigma(x)\, n_\mu(x) \overline{\psi}(x) \gamma^\mu \varphi(x) \tag{2}$$

is well-defined for $\psi, \varphi \in L^2(\Sigma, \mathbb{C}^4)$ and defines a scalar product on that space. Use this result to conclude that $\mathscr{H}_\Sigma = (L^2(\Sigma, \mathbb{C}^4), \langle \cdot, \cdot \rangle_\Sigma)$ is a Hilbert space.

Exercise 3. *Probability conservation via propagator methods.*

Assume that for every hyperplane $\Sigma \subset \mathbb{M}$ and every $\psi_\Sigma \in \mathcal{S}(\Sigma)^4$ (the space of four-component test functions on Σ), the initial value problem $\psi(x) = \psi_\Sigma(x)$, $x \in \Sigma$ has a unique solution $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ given by

$$\psi(x) = -i \int_\Sigma d\sigma(y)\, S(x-y) \gamma^\mu n_\mu(y) \psi_\Sigma(y). \tag{3}$$

Here, S denotes the propagator of the Dirac equation (see lecture).

(a) Show that $S(0, \mathbf{x}) = i\gamma^0 \delta^{(3)}(\mathbf{x})$.

(b) Demonstrate that $-i \int_\Sigma S(x-y)\gamma^\mu n_\mu(y) S(y-z) = S(x-z)$ for every space-like hyperplane Σ.

(c) Using $\gamma^0 S^\dagger(x) \gamma^0 = -S(-x)$, show an analogous integral formula for $\overline{\psi}(x)$.

(d) Using only (3) and (c), show that for every pair of space-like hyperplanes $\Sigma_1, \Sigma_2 \subset \mathbb{M}$, we have (probability conservation):

$$\int_{\Sigma_1} d\sigma(x)\, n_{1,\mu}(x)\, \overline{\psi}(x) \gamma^\mu \psi(x) = \int_{\Sigma_2} d\sigma(x)\, n_{2,\mu}(x)\, \overline{\psi}(x) \gamma^\mu \psi(x). \tag{4}$$