Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 11

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Exercise 1. *Free multi-time Dirac equations.*

Let $N \in \mathbb{N}$ and consider the equations

$$(i\gamma_k^\mu \partial_{k,\mu} - m_k)\psi(x_1, ..., x_N) = 0, \quad k = 1, ..., N \tag{1}$$

for the multi-time wave function

$$\psi : \mathbb{R}^{4N} \to (\mathbb{C}^4)^{\otimes N} \cong \mathbb{C}^{4^N}. \tag{2}$$

(a) Show that the initial value problem $\psi(0, \mathbf{x}_1, ..., 0, \mathbf{x}_N) = \psi_0(\mathbf{x}_1, ..., \mathbf{x}_N)$, $\mathbf{x}_1, ..., \mathbf{x}_N \in \mathbb{R}^3$ for (1) with $\psi_0 \in \mathcal{S}(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$ has a solution $\psi \in C^1(\mathbb{R}^{4N}, \mathbb{C}^{4^N})$.

Hint: Show that each of the N equations (1) can be solved separately.

(b) Let $\psi \in C^1(\mathbb{R}^{4N}, \mathbb{C}^{4^N})$ be a solution of (1) and let

$$\overline{\psi} = \psi^\dagger \gamma_1^0 \cdots \gamma_N^0 \tag{3}$$

Show that the Dirac tensor current

$$j^{\mu_1 ... \mu_N}(x_1, ..., x_N) = \overline{\psi}(x_1, ..., x_N)\gamma_1^{\mu_1} \cdots \gamma_N^{\mu_N} \psi(x_1, ..., x_N) \tag{4}$$

satisfies the continuity equations

$$\partial_{k,\mu_k} j^{\mu_1 ... \mu_k ... \mu_N}(x_1, ..., x_N) = 0, \quad k = 1, ..., N. \tag{5}$$

(c) Show that as a consequence of (5) the total probability integral

$$P(\Sigma) = \int_\Sigma d\sigma(x_1) \cdots \int_\Sigma d\sigma(x_N)\, j^{\mu_1 ... \mu_N}(x_1, ..., x_N)\, n_{\mu_1}(x_1) \cdots n_{\mu_N}(x_N) \tag{6}$$

satisfies $P(\Sigma) = P(\Sigma')$ for all space-like hyperplanes $\Sigma, \Sigma' \subset \mathbb{M}$.

(d) Let $\psi_1, \psi_2 \in C^1(\mathbb{R}^{4N}, \mathbb{C}^{4^N})$ both be solutions of the same initial value problem as in (a). Assume in addition that $\psi(x_1, ..., x_N)$ is square-integrable in each spatial variable $\mathbf{x}_i$. Using the technique of energy integrals for $P(\Sigma)$, show that $\psi_1 = \psi_2$ on all space-like hyperplanes.

Exercise 2. *Space-like configurations generalize non-collision configurations.*

(a) In the case of $N = 2$ particles, show that the set of space-like configurations

$$\mathscr{S} = \{(x_1, x_2) \in \mathbb{R}^4 \times \mathbb{R}^4 : (x_1 - x_2)_\mu (x_1 - x_2)^\mu < 0\}. \tag{7}$$

is the smallest Poincare invariant set which contains the equal-time collision-free configurations

$$C = \{(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) \in \mathbb{R}^4 \times \mathbb{R}^4 \,|\, t_1 = t_2 \text{ and } \mathbf{x}_1 \neq \mathbf{x}_2\}. \tag{8}$$

Note: A Poincare transformation (a, Λ) acts on $(x_1, x_2) \in \mathbb{R}^4 \times \mathbb{R}^4$ as $(\Lambda x_1 + a, \Lambda x_2 + a)$.

(b) In the case of $N = 2$, find at least three different Poincare invariant non-empty proper subsets of $\mathbb{R}^{4N}$.
Optional question: How many different such subsets are there for $N = 2$?