Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 12

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Exercise 1. *No-go theorem for potentials in multi-time equations with Laplacians*

Consider the multi-time system

$$\begin{aligned} i\partial_{t_1}\psi &= (-\Delta_1 + V_1(\mathbf{x}_1, \mathbf{x}_2))\psi, \\ i\partial_{t_2}\psi &= (-\Delta_2 + V_2(\mathbf{x}_1, \mathbf{x}_2))\psi \end{aligned} \tag{1}$$

for a multi-time wave function $\psi : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C}$. Here, Δ_i denotes the Laplacian with respect to $\mathbf{x}_i$, $i = 1, 2$ and $V_1, V_2 : \mathbb{R}^6 \to \mathbb{R}$ are smooth functions.

(a) State the appropriate consistency condition for this case.

(b) Show that this consistency condition is only satisfied if V_1 does not depend on $\mathbf{x}_2$ and V_2 does not depend on $\mathbf{x}_1$.

(c) What does this result mean physically?

Exercise 2. *The current density form ω_j*

For every $\mu, \nu = 0, 1, 2, 3$ let

$$j^{\mu\nu} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (x_1, x_2) \mapsto j^{\mu\nu}(x_1, x_2) \tag{2}$$

be a C^1-function with compact support in the spatial variables $\mathbf{x}_1$, $\mathbf{x}_2$.

For $N = 2$ particles, the *current density form* ω_j is defined to be the differential form (a 6-form on $\mathbb{R}^8$)

$$\omega_j = \sum_{\mu,\nu=0}^{3} (-1)^{\mu+\nu} j^{\mu\nu} (dx_1^0 \wedge \cdots \wedge \widehat{dx_1^\mu} \wedge \cdots \wedge dx_1^3) \wedge (dx_2^0 \wedge \cdots \wedge \widehat{dx_2^\nu} \wedge \cdots \wedge dx_2^3) \tag{3}$$

where $\widehat{(\cdot)}$ denotes omission.

(a) Let $\Sigma \subset \mathbb{M}$ be a spacelike hyperplane with future-pointing unit normal vector field n. Use the identity

$$n_\mu d\sigma = (-1)^\mu \, dx^0 \wedge \cdots \wedge \widehat{dx^\mu} \wedge \cdots \wedge dx^3 \tag{4}$$

to show that the integral

$$P(\Sigma) = \int_\Sigma d\sigma(x_1) \int_\Sigma d\sigma(x_2) \, n_\mu(x_1) n_\nu(x_2) j^{\mu\nu}(x_1, x_2) \tag{5}$$

can be rewritten as

$$P(\Sigma) = \int_{\Sigma\times\Sigma} \omega_j. \tag{6}$$

(b) Assuming that $j^{\mu\nu}$ satisfies the continuity equations

$$\partial_{1,\mu} j^{\mu\nu}(x_1, x_2) = 0, \quad \nu = 0,1,2,3 \quad \text{and} \quad \partial_{2,\nu} j^{\mu\nu}(x_1, x_2) = 0, \quad \mu = 0,1,2,3, \tag{7}$$

show that the exterior derivative $d\omega_j$ of ω_j vanishes.

(c) Given (7), show that $P(\Sigma) = P(\Sigma')$ for all pairs of space-like hyperplanes Σ, Σ'. *Hint:* Use the theorem of Stokes for ω_j together with the result of (b).

(d) Write down the analog of ω_j for 1+1 spacetime dimensions and convince yourself that the previous points also hold then.

(e) In 1+1 spacetime dimensions, let

$$\mathscr{S}_1 := \{(t_1, z_1, t_2, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : (t_1 - t_2)^2 - (z_1 - z_2)^2 < 0 \text{ and } z_1 < z_2\}, \tag{8}$$

i.e., the set of space-like configurations with $z_1 < z_2$. Using Stokes' theorem, extract a condition on the tensor current $j^{\mu\nu}$ such that

$$\int_{(\Sigma\times\Sigma)\cap\mathscr{S}_1} \omega_j = \int_{(\Sigma'\times\Sigma')\cap\mathscr{S}_1} \omega_j \tag{9}$$

holds for all pairs of space-like hyperplanes $\Sigma, \Sigma' \subset \mathbb{R}^2$.

Hint: The condition on $j^{\mu\nu}$ should be a linear condition between the components of j which needs to hold on a subset of the boundary $\partial\mathscr{S}_1$ of $\mathscr{S}_1$.

(f) Let $j^{\mu\nu} = \overline{\psi}\gamma_1^\mu\gamma_2^\nu\psi$ where ψ solves the free multi-time Dirac equations. What is the physical meaning of (9)?