Exercises for "Wave Equations of Relativistic Quantum Mechanics"

Sheet 13

Winter Semester 2018/19

Dr. Matthias Lienert

Exercise 1. *Contact interactions for a multi-time wave function in 1+1 dimensions.*

Consider the multi-time model in 1+1 spacetime dimensions defined by

$$\psi : \overline{\mathscr{S}}_1 \to \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4 \tag{1}$$

with $\mathscr{S}_1 = \{(t_1, z_1, t_2, z_2) \in \mathbb{R}^4 : |t_1 - t_2| < |z_1 - z_2| \text{ and } z_1 < z_2\}$,
multi-time equations

$$i\gamma_k^\mu \partial_{k,\mu} \psi(x_1, x_2) = 0, \quad k = 1, 2, \tag{2}$$

where $x_i = (t_i, z_i) \in \mathbb{R}^2$ and $\gamma^0 = \sigma^1$, $\gamma^1 = \sigma^1\sigma^3$,
initial conditions

$$\psi(0, z_1, 0, z_2) = g(z_1, z_2), \quad z_1 < z_2 \tag{3}$$

for some $g \in C_c^1(\mathbb{R}^2, \mathbb{C}^4)$,
and the boundary condition

$$\psi_2(t, z, t, z) = e^{i\theta}\psi_3(t, z, t, z), \quad t, z \in \mathbb{R} \tag{4}$$

for some $\theta \in [0, 2\pi)$.

Prove that this model has the unique solution

$$\begin{aligned}
\psi_1(t_1, z_1, t_2, z_2) &= g_1(z_1 - t_1, z_2 - t_2), \\
\psi_2(t_1, z_1, t_2, z_2) &= \begin{cases} g_2(z_1 - t_1, z_2 + t_2) & \text{for } z_1 - t_1 < z_2 + t_2, \\ e^{i\theta} g_3(z_2 + t_2, z_1 - t_1) & \text{for } z_1 - t_1 \geq z_2 + t_2, \end{cases} \\
\psi_3(t_1, z_1, t_2, z_2) &= \begin{cases} g_3(z_1 + t_1, z_2 - t_2) & \text{for } z_1 + t_1 < z_2 - t_2, \\ e^{-i\theta} g_2(z_2 - t_2, z_1 + t_1) & \text{for } z_1 + t_1 \geq z_2 - t_2, \end{cases} \\
\psi_4(t_1, z_1, t_2, z_2) &= g_4(z_1 + t_1, z_2 + t_2)
\end{aligned} \tag{5}$$

with $\psi \in C^1(\overline{\mathscr{S}}_1, \mathbb{C}^4)$ provided that for all $z \in \mathbb{R}$:

(i) $g_2(z, z) = e^{i\theta} g_3(z, z)$,

(ii) $(\partial_1 g_2)(z, z) = e^{i\theta}(\partial_2 g_3)(z, z)$,

(iii) $(\partial_2 g_2)(z, z) = e^{i\theta}(\partial_1 g_3)(z, z)$.