Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Anna Sancassani Wintersemester 19/20

Groups and Representations

Homework Assignment 3 (due on 6 Nov 2019)

Problem 12
Let $(\mathbb{R} \setminus \{0\}, \cdot)$ be the multiplicative group of real numbers, and

$$\Gamma(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \setminus \{0\},$$

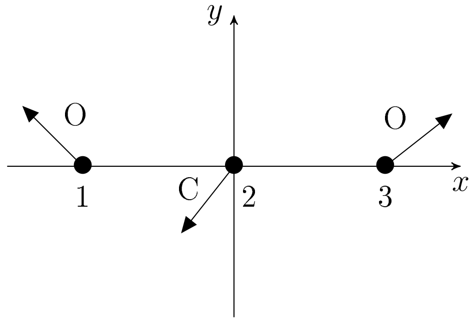
a representation on $\mathbb{C}^2$. Find all invariant subspaces. Is Γ completely reducible?

Problem 13
Let $\varphi : G \to \mathrm{U}(n)$ be a unitary irreducible representation of a group. Show: If G is abelian then $n = 1$.

Problem 14
Let G be a finite group and $\Gamma : G \to \mathrm{GL}(V)$ a finite dimensional representation. Prove that $|\det \Gamma(g)| = 1 \; \forall \, g \in G$.

Problem 15
CO_2 is a linear molecule; in its ground state the carbon atom sits in the middle between the two oxygen atoms. The symmetry group of this system is isomorphic to the Klein four group $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and has the following elements: the identity (e), reflections (σ_x and σ_y) across the x- and y-axis, respectively, and a rotation (R) by 180° about the origin.

A coplanar vibration entails displacements of the 3 atoms in a fixed plane. It can be characterised by a vector $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$.



Determine the action of the symmetry group on the canonical basis of $\mathbb{R}^6$. Write down the resulting six dimensional representation of V_4. Is this representation irreducible?

Problem 16
Let D_4 be the symmetry group of a square. We denote by R the rotation by $\frac{\pi}{2}$ and by σ the reflection across the diagonal through the lower left and upper right corner. We write all group elements as $R^k\sigma^\ell$ for some k and ℓ. (Why is this possible and which values do k and ℓ take?)

a) Find all conjugacy classes.
HINT: Determine $\sigma R \sigma$ first, this simplifies calculations a lot.

b) Determine all normal subgroups and the isomorphism types of the corresponding quotient groups (i.e. name known groups to which they are isomorphic).

c) Is D_4 isomorphic to a direct product of non-trivial subgrous?

Problem 17 (Continuation of Problem 11)
Let $\Lambda \in \mathrm{O}(3,1)$ be time orientation preserving, i.e. $d(e_0, \Lambda e_0) > 0$. Show that there exist $U, V \in \mathrm{O}(3)$ and a boost B in x_3-direction, such that

$$\Lambda = UBV\,.$$

HINT: First consider Λe_0 and find U and B such that $B^{-1}U^{-1}\Lambda e_0 = e_0$.