Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Anna Sancassani Wintersemester 19/20

Groups and Representations

Homework Assignment 5 (due on 20 Nov 2019)

Problem 23
We consider a rotationally invariant Hamiltonian. Let E be an eigenvalue of H with eigenspace V_E spanned by the spherical harmonics $Y_{1m}(\varphi, \vartheta) = \cos\vartheta \mathrm{e}^{\mathrm{i}m\varphi}$ with a fixed radial part R, i.e. $V_E = \mathrm{span}(\{R(r)Y_{1m}(\varphi, \vartheta) : m = -1, 0, 1\})$.[2]

V_E carries a three-dimensional irreducible representation of O(3), defined by $(\Gamma(U)\psi)(x) = \psi(U^{-1}x)$. O(3) contains the subgroup $D_3 = \{e, C, \bar{C}, \sigma_1, \sigma_2, \sigma_3\} \cong S_3$, where C and $\bar{C}$ denote rotations about the z-axis (cf. Section 2.4.1).

Study the effect of perturbations that are only invariant under D_3 or $\mathbb{Z}_3 \cong \{e, C, \bar{C}\}$. Determine the relevant irreducible representations with their multiplicities and sketch the possible splitting of energy levels.

Problem 24
We consider once more the CO_2 molecule of Problem 15.

a) How many non-equivalent irreps does the symmetry group V_4 have, and what are their dimensions?

b) Determine the character table for V_4.

In Problem 15 we found a six-dimensional representation of V_4.

c) Which irreps are contained in this six-dimensional representation?

d) Decompose the six-dimensional carrier space into irreducible invariant subspaces by applying the generalised projection operators.

Problem 25
$V = \mathbb{C}^2$ carries the 2-dimensional irreducible representation of $D_3 \cong S_3$ (cf. Section 2.4.1). On $W = V \otimes V$ we consider the corresponding product representation. Decompose W into irreducible invariant subspaces by applying the generalised projection operators, and determine the Clebsch-Gordan coefficients.

---

[2] We use spherical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}.$$