Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 6 (due on 27 Nov 2019)

Problem 26
We consider the abelian group $C_3 = \{e, a, a^{-1}\} \cong \mathbb{Z}_3$.

a) How many (non-equivalent) irreps does C_3 have, what are their dimensions and how often do they appear in the regular rep?

b) Show that
$$e_1 = \frac{1}{3}(e + a + a^{-1})$$
is a primitive idempotent, generating the trivial rep.

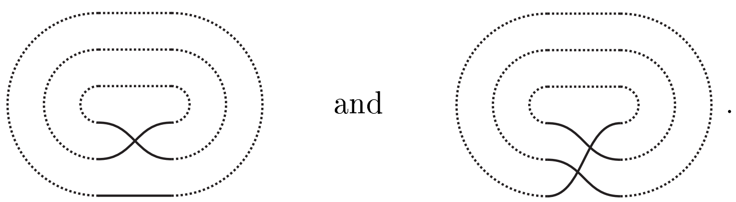
c) Use the ansatz
$$e_2 = xe + ya + za^{-1}$$
in order to find all primitive idempotents.

d) For each primitive idempotent find out whether it generates a new (non-equivalent) irrep or an irrep equivalent to one generated by a previous idempotent.

e) Specify all minimal left ideals and construct the corresponding irreps of C_3. Collect your results in a table.

Problem 27
For $\sigma \in S_n$ and $j = 1, \ldots, n$ let $k_j(\sigma)$ be the number of (disjoint) cycles of length j in σ, e.g. $k_1(e) = n$ and $k_j(e) = 0 \;\forall\, j > 1$. Show:

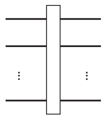
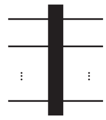
a) The conjugacy class of σ is determined by its cycle structure, i.e.
$$[\sigma] := \{\tau\sigma\tau^{-1} : \tau \in S_n\} = \{\tau \in S_n : k_j(\tau) = k_j(\sigma), j = 1, \ldots, n\}\,.$$
It's almost trivial using the birdtrack notation (see Section 1.4)!
HINT: In order to make the cycle structure visible consider the birdtrack diagram of σ and connect the first line on the left to the first line on the right etc.; e.g. for $(12), (132) \in S_3$ consider



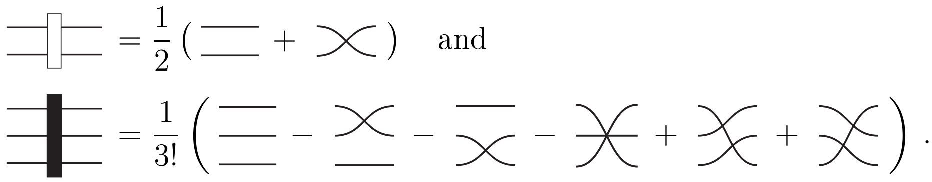
b) The number of elements of a class is given by
$$|[\sigma]| = \frac{n!}{\prod\limits_{j\le n} k_j!\, j^{k_j}}\,.$$

Problem 28

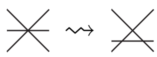
In birdtrack notation we denote symmetrisers and anti-symmetrisers by open and solid bars, respectively, i.e.

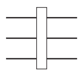
$\frac{1}{n!}s = \frac{1}{n!}\sum_{p\in S_n} p =$  and $\frac{1}{n!}a = \frac{1}{n!}\sum_{p\in S_n} \mathrm{sgn}(p) =$  .

Note that we include a factor of $\frac{1}{n!}$ in the definition of bars over n lines. For instance,

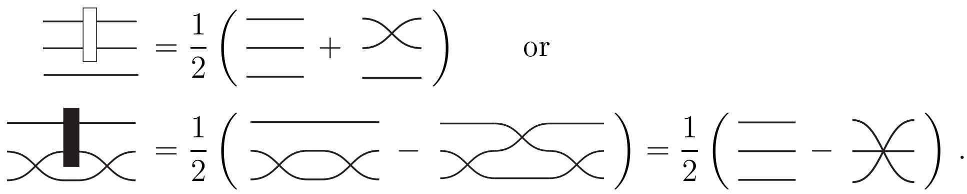


(∗)

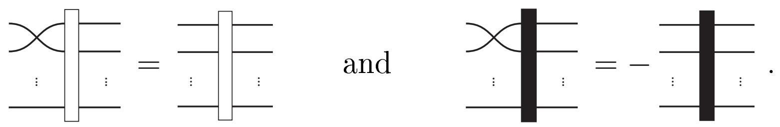
Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  (K=3).

a) Expand  and  as in (∗).

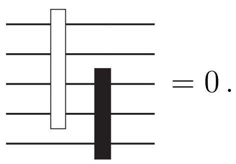
We also use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.



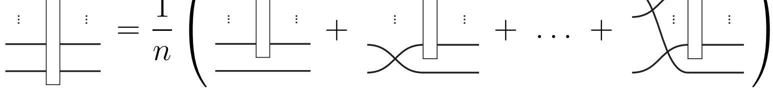
It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.

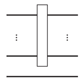


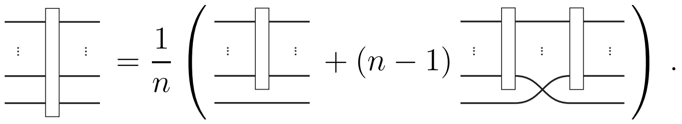
b) Explain why this implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.



Symmetrisers and anti-symmetrisers can be built recursively. To this end notice that on the r.h.s. of



we have sorted the terms according to where the last line is mapped – to the nth, to the $(n-1)$th, $\ldots$, to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation



c) Derive the corresponding recursion relation for anti-symmetrisers.