

Groups and Representations

Homework Assignment 7 (due on 4 Dec 2019)

Problem 29

Verify that the Young operators for the standard Young tableaux for S_3 (see Section 5.3) are essentially idempotent and primitive and add up to the identity. (Use cycle and birdtrack notation for some of your calculations.) Determine once more the characters of the irreps of S_3 by using the methods of Section 4.3.1.

Problem 30

Let V be a vector space and $A : V \rightarrow V$ a linear map. Show that if A is nilpotent (i.e. if for some $n \in \mathbb{N}$ we have $A^n v = 0 \forall v \in V$) then $\text{tr } A = 0$.

Problem 31

For a fixed partition λ of $n \in \mathbb{N}$ we define an ordering of standard tableaux as follows. Consider the sequence $(r_\lambda^p)_j, j = 1, \dots, n$, of numbers in the boxes of Θ_λ^p starting with the first row read from left to right, then the second row from left to right etc. We say that $\Theta_\lambda^p > \Theta_\lambda^q$ if the first non-vanishing term in the sequence $(r_\lambda^p)_j - (r_\lambda^q)_j, j = 1, \dots, n$, is positive. Then, e.g., the standard tableaux for $\lambda = (3, 2)$ are ordered as

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} < \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} .$$

- Prove that $e_\lambda^p e_\lambda^q = 0$ if $\Theta_\lambda^p > \Theta_\lambda^q$.
- Show that (a) implies that the left ideals generated by the standard tableaux for a fixed partition are linearly independent.

Problem 32

In the quark model baryons are made out of three quarks. The latter are characterised i.a. by the quantum numbers I (isospin) and Y (hyper charge).

We have $(I, Y) = (\frac{1}{2}, \frac{1}{3})$ for the up-quark, $|u\rangle$, $(I, Y) = (-\frac{1}{2}, \frac{1}{3})$ for the down-quark, $|d\rangle$, and $(I, Y) = (0, -\frac{2}{3})$ for the strange-quark, $|s\rangle$. For products like $|udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$ the values of I and Y are given by the sums of the values for the individual quarks.

For combinations of 3 quarks we thus have a 27-dimensional space V , which carries a representation of S_3 (by permutation of the factors).

- Which irreps are contained in this representation and what are their multiplicities?
- Let $U \subset V$ be an irreducible invariant subspace. What can we say about the values of I and Y on U ?
- In a (I, Y) -diagram mark all points corresponding to vectors transforming in the irrep defined by $\square\square\square$.
- Repeat part (c) for the irrep with Young diagram $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$. You find some potentially useful `Octave`/`MATLAB`-Code on the course webpage.

Problem 33

For $A \in \mathbb{C}^{n \times n}$ the matrix exponential is defined as

$$e^A = \exp(A) = \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!}.$$

Prove:

- The series converges absolutely and uniformly.
HINT: On $\mathbb{C}^{n \times n}$ use the operator norm

$$\|A\| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} \frac{|Av|}{|v|},$$

for which we have $\|AB\| \leq \|A\| \|B\|$.

- For $T \in \text{GL}(n)$ we have $e^{TAT^{-1}} = Te^AT^{-1}$.
- e^{tA} is the unique solution of the initial value problem $\dot{X}(t) = AX(t)$, $X(0) = \mathbf{1}$.
- For $t, s \in \mathbb{C}$ we have $e^{(t+s)A} = e^{tA}e^{sA}$.
- $(e^A)^\dagger = e^{(A^\dagger)}$.
- $\det e^A = e^{\text{tr } A}$.