Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Anna Sancassani Wintersemester 19/20

Groups and Representations

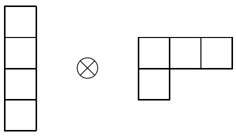
Homework Assignment 8 (due on 11 Dec 2019)

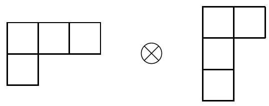
Problem 34

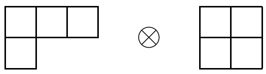
a) Determine the dimensions of all irreps of S_4 using the methods of Section 5.5 (hook rule).

b) Determine the character table of S_4 using the methods of Section 5.5 (recursive method).

c) Consider the following product representations of S_4, determine which irreps they contain and how many times. Also write their decomposition into irreps in terms of Young diagrams.

(i) 

(ii) 

(iii) 

(iv) 

Problem 35

Let $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ be a unit vector in $\mathbb{R}^3$ and $\varphi \in \mathbb{R}$. We denote by σ_j, $j = 1, 2, 3$, the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we define

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} .$$

Show that

$$\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) = \mathbb{1}\cos\tfrac{\varphi}{2} - \mathrm{i}\vec{\sigma}\cdot\vec{n}\sin\tfrac{\varphi}{2} ,$$

and verify that $\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) \in \mathrm{SU}(2)$.

Hint: First calculate $(\vec{\sigma}\cdot\vec{n})^2$.

Problem 36

We define $\mathfrak{sl}(2,\mathbb{C}) := \{A \in \mathbb{C}^{2\times 2} : \operatorname{tr} A = 0\}$. Then the matrix exponential is a map

$$\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C}) = \{B \in \mathbb{C}^{2\times 2} : \det B = 1\} .$$

a) Show that the matrix

$$S_a = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

is in the image of exp iff $a = 0$.

b) Is $\mathrm{SL}(2,\mathbb{C})$ compact?