Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 9 (due on 18 Dec 2019)

Problem 37
The Lie algebra of SU(2) is the (real) vector space

$$\mathfrak{su}(2) = \{X \in \mathbb{C}^{2\times 2} : \mathrm{tr}(X) = 0, X^\dagger = X\}\,.$$

A basis is given by the Pauli matrices (see Problem 35). Show:

a) SU(2) acts on $\mathfrak{su}(2)$ by conjugation: $X \mapsto UXU^\dagger$.

b) $\langle X, Y\rangle := \frac{1}{2}\,\mathrm{tr}(XY)$ defines a scalar product on $\mathfrak{su}(2)$.
HINT: Begin by calculating $\mathrm{tr}(\sigma_i\sigma_j)$.

c) Every $U \in \mathrm{SU}(2) \cong S^3$ (cf. Problem 22) can be written as $\mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\vec{\sigma}\cdot\vec{n}}$ with $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ (cf. Problem 35). Over which values does α run?

Problem 38
The elements of $\mathfrak{su}(2)$ can be written as $X = \vec{\sigma}\cdot\vec{x}$ with $\vec{x} \in \mathbb{R}^3$ (cf. Problems 35 & 37). The action of SU(2) on $\mathfrak{su}(2)$ by conjugation (see Problem 37) then defines a homomorphism

$$\begin{aligned}\varphi : \mathrm{SU}(2) &\to \mathrm{GL}(3,\mathbb{R})\\ \vec{\sigma}\cdot\varphi(U)\vec{x} &:= U(\vec{\sigma}\cdot\vec{x})U^\dagger\,.\end{aligned}$$

Show that

a) $\varphi(U)_{ij} = \frac{1}{2}\,\mathrm{tr}(\sigma_i U\sigma_j U^\dagger)$,

b) $\varphi(U)^T = \varphi(U)^{-1}$, and

c) $\det(\varphi(U)) = 1$. HINT: Recall the connectedness properties of SU(2).

Hence $\varphi(\mathrm{SU}(2)) \subset \mathrm{SO}(3)$.

d) Determine the kernel of φ.

e) Calculate $\varphi(U_\alpha)$ for $U_\alpha = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\sigma_3}$, $\alpha \in [0, 4\pi)$ and explain that $\varphi(\mathrm{SU}(2)) = \mathrm{SO}(3)$.
What can we now conclude using the homomorphism theorem (Problem 10)?

Problem 39

Let V be a (complex, finite dimensional) vector space and let V^* be its dual, i.e. the space of all linear maps $V \to \mathbb{C}$. For a linear map $A : V \to V$ we define its dual $A^* : V^* \to V^*$ by $V^* \ni f \mapsto A^*(f) := f \circ A$. Let G be group and $\Gamma : G \to \mathrm{GL}(V)$ a representation.

a) Define a representation $\Gamma^* : G \to \mathrm{GL}(V^*)$ in a natural way.

   HINT: Simply replacing $\Gamma(g) : V \to V$ by its dual map doesn't quite work (why?) but with a slight modification it does.

Let $\{e_j\}$ be a basis of V and $\{f_j\}$ the corresponding dual basis, i.e. $f_j(e_k) = \delta_{jk}\ \forall\, j, k = 1, \ldots, \dim V = \dim V^*$. For $g \in G$ we express $\Gamma(g) : V \to V$ and $\Gamma^*(g) : V^* \to V^*$ as matrices in the bases $\{e_j\}$ and $\{f_j\}$, respectively.

b) What is the relation between these two matrices? What happens if Γ is unitary?