Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 12 (due on 22 Jan 2020)

Problem 44

Let $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form from Problem 43, and let G be such that K is positive definite. We choose an orthonormal basis $\{X_j\}$ with respect to K, i.e. $K(X_j, X_k) = \delta_{jk}$, and define $C_2 \in E(\mathfrak{g})$ by

$$C_2 := \sum_j X_j X_j \, .$$

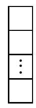
Show:

a) C_2 is independent of the choice of basis.

b) C_2 is a Casimir operator (the so-called *quadratic Casimir operator*), i.e.

$$\mathrm{Ad}_g(C_2) = C_2 \quad \forall\, g \in G \, .$$

Problem 45

We show that the $\mathrm{GL}(N)$ irrep corresponding to the Young diagram $\Theta_\mathrm{a} =$  with N rows is given by the determinant:

- First recall that for vectors $|i_1, \ldots, i_N\rangle$ contributing to $e_\mathrm{a} g|\alpha\rangle$ all i_k are different.
- Write these vectors as $p|1, \ldots, N\rangle$ with a permutation p.
- Then calculate $e_\mathrm{a} g|1, \ldots, N\rangle$ for $g \in \mathrm{GL}(N)$.

Which irrep corresponds to Θ_a if we replace $\mathrm{GL}(N)$ by the subgroup $\mathrm{SU}(N) \subset \mathrm{GL}(N)$?