Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 14 (due on 5 Feb 2020)

Problem 49
Within a hadron multiplet masses of particles with the same isospin are approximately equal, whereas mass differences between different isospin multiplets (within the same hadron multiplet) are larger. This is due to the strange-quark being significantly heavier than the up-quark and the down-quark which explicitly breaks the $\mathrm{SU}(3)_{\text{flavor}}$-symmetry. Since the strong interaction conserves isospin I and hypercharge Y the remaining symmetry group is $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$.

In this problem we derive the Gell-Mann-Okubo mass formula for the baryon decuplet, which in 1961 allowed to predict the mass of the then unknown Ω^--particle.

We assume that the Hamiltonian is of the form $H = H_0 + H'$, with H_0 invariant under $\mathrm{SU}(3)_{\text{flavor}}$ and a small perturbation H' which is still invariant under $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$. Each baryon is described by a normalised state $|\psi\rangle$ with mass $m = \langle\psi|H|\psi\rangle$.

The unperturbed states, denoted as $\psi_i^\lambda \equiv \psi_{(I_3Y)I}^\lambda$, form SU(3)-multiplets, with λ labelling SU(3)-irreps. If we ignore the perturbation H' then we obtain the same mass $\langle\psi_i^\lambda|H_0|\psi_i^\lambda\rangle = a_\lambda$ for all states within a given SU(3)-multiplet. In first order perturbation theory mass differences within a multiplet are given by the eigenvalues Δm_i^λ of the matrix $M_{ij}^\lambda = \langle\psi_i^\lambda|H'|\psi_j^\lambda\rangle$. Due to conservation of isospin and hypercharge this matrix is diagonal, i.e.

$$\Delta m_i^\lambda = \langle\psi_i^\lambda|H'|\psi_i^\lambda\rangle\ .$$

a) We assume that H' is a linear combination of irreducible operators (see Section 4.2) with respect to SU(3). (Why is this a reasonable assumption?) Show that the singlet part of this expansion does not lead to mass differences within a multiplet but only to an overall shift which can be absorbed in a_λ.

b) Explain why the invariance of H' under $\mathrm{SU}(2)_i \times \mathrm{U}(1)_Y$ means that H' can only contain operators O_k^μ with $k = (00)0$, i.e. with $Y = I = I_3 = 0$.

   Show that thus the triplet and decuplet reps (cf. Problems 32 & 48) cannot contribute to H'.

   We remark that the sextet $\ydiagram{2}$ does not contribute to H' either. If we neglect higher dimensional irreps then H' can only contain operators transforming like $O_{(00)0}^8$.

c) Consider now $\Delta m_i^\lambda = \langle\psi_i^\lambda|O_{(00)0}^8|\psi_i^\lambda\rangle$. Here $O_{(00)0}^8|\psi_i^\lambda\rangle$ transforms in the product rep $\Gamma^\lambda \otimes \ydiagram{2,1}$. Explain why for Δm_i^λ only the Γ^λ-part (in the decomposition) of this product is relevant.

d) Formulate the Wigner-Eckart Theorem for $\langle\psi_i^\lambda|O_{(00)0}^8|\psi_i^\lambda\rangle$ and use the result of Problem 47 in order to determine the number of free parameters (reduced matrix elements) for a given irrep λ.

e) Let now Γ^λ be an irrep corresponding to a rectangular Young diagram, and let S, T be two operators transforming like $O^8_{(00)0}$. Does $\langle\psi_i^\lambda|S|\psi_i^\lambda\rangle/\langle\psi_i^\lambda|T|\psi_i^\lambda\rangle$ depend on I or Y? Explain using part (d).

f) The 8 generators of SU(3), i.e. a basis for $\mathfrak{su}(3)$, can be chosen as the Gell-Mann matrices (here normalised s.t. $\mathrm{tr}(\lambda_j\lambda_k) = 2\delta_{jk}$)

$$\lambda_i = \begin{pmatrix} \multicolumn{2}{c}{\sigma_i} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } i = 1,2,3, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They transform in the adjoint or octet rep, corresponding to $\ydiagram{2,1}$. The first three matrices, $\lambda_1, \lambda_2, \lambda_3$ generate $\mathrm{SU}(2)_I$. Show that $\frac{1}{\sqrt{3}}\lambda_8$ transforms like $O^8_{(00)0}$ and explain why implies

$$\Delta m_i^\lambda = b_\lambda\, Y \quad \text{for rectangular Young diagrams } \Theta_\lambda.$$

In particular this is true for the decuplet with $\Theta_\lambda = \ydiagram{3}$.

g) The following table shows masses (in MeV/c^2, according to C. Patrignani et al. (Particle Data Group), Chin. Phys. C **40** (2016) 100001 and 2017 update) and some quantum numbers for nine of the ten particles in the baryon-decuplet ($J^P = \frac{3}{2}^+$). Which mass should we expect for the missing particle?

	Δ^-	Δ^0	Δ^+	Δ^{++}	Σ^{*-}	Σ^{*0}	Σ^{*+}	Ξ^{*-}	Ξ^{*0}
I	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$
I_3	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	-1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$
m	1232	1232	1232	1232	1387	1384	1383	1535	1532