Mathematical Quantum Theory
Exercise sheet 1
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Exercise 1. The goal of this exercise is to describe the evolution of a simple two-state system. Let $\mathcal{H} = \mathbb{C}^2$ be the Hilbert space. A basis for this Hilbert space is given by the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ("spin up" state), $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ("spin down" state). Consider the Hamiltonian:

$$H = -B\sigma_x \ , \qquad \text{where } \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B \in \mathbb{R}. \tag{1}$$

Consider a quantum state described at $t = 0$ by the wave function

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{2}$$

Let ψ_t be the solution of the Schrödinger equation:

$$i\partial_t \psi_t = H\psi_t \ , \qquad \psi_0 = \psi \ . \tag{3}$$

(a) Compute ψ_t for all times.

(b) Let $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The eigenvectors of σ_z are the "spin up" and "spin down" states. Suppose that at the time t we perform a measurement of σ_z on ψ_t: compute the probability for finding a "spin down" state.

(c) Consider the variances of the observables σ_x, σ_z on the state ψ_t:

$$\Delta\sigma_x^{\psi_t} = \langle \psi_t, (\sigma_x - \langle \psi_t, \sigma_x \psi_t \rangle)^2 \psi_t \rangle \ , \qquad \Delta\sigma_z^{\psi_t} = \langle \psi_t, (\sigma_z - \langle \psi_t, \sigma_z \psi_t \rangle)^2 \psi_t \rangle \ . \tag{4}$$

Find the values of t for which the Heisenberg's uncertainty principle is saturated.

Exercise 2. Prove the following relations between $L^2(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$.

(a) Neither the inclusion $L^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ nor the inclusion $L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ is true.
[Hint. Consider the functions $|x|^{-\alpha}$, for $|x| \le 1$ or for $|x| > 1$.]

(b) Suppose that $f \in L^2(\mathbb{R}^d)$ is supported in a ball $\{x \in \mathbb{R}^d \mid |x| \le R\}$. Then, f is in $L^1(\mathbb{R}^d)$.
[Hint. Recall the Cauchy-Schwarz inequality.]

(c) Suppose that f is bounded: $|f(x)| \le M$, and that $f \in L^1(\mathbb{R}^d)$. Then, $f \in L^p(\mathbb{R}^d)$ for all $p \ge 1$.

Exercise 3. Compute the Fourier transform of $g_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$:

$$g_\varepsilon(x) = e^{-\varepsilon |x|^2} \ , \qquad \varepsilon > 0 \ . \tag{5}$$