Mathematical Quantum Theory
Exercise sheet 2
05.11.2019

Giovanna Marcelli
giovanna.marcelli@uni-tuebingen.de

Exercise 1. Consider the two-level system of the previous exercise sheet. Let $\sigma_x, \sigma_y, \sigma_z$ the *Pauli matrices:*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1}$$

Let $\hat{S}$ be the *spin operator:*

$$\hat{S} := (\hat{S}_x, \hat{S}_y, \hat{S}_z) , \qquad \hat{S}_i = \frac{1}{2}\sigma_i , \qquad i = x, y, z . \tag{2}$$

Given an observable O, its Heisenberg evolution is defined as:

$$O(t) := e^{iHt} O e^{-iHt} . \tag{3}$$

To understand the meaning of the Heisenberg evolution of O, simply notice that:

$$\langle \psi, O(t)\psi \rangle = \langle \psi_t, O\psi_t \rangle . \tag{4}$$

(a) Compute the Heisenberg evolution of the spin operator:

$$\hat{S}(t) = e^{iHt}\hat{S}e^{-iHt} , \qquad H = -B\sigma_x , \qquad B \in \mathbb{R} . \tag{5}$$

(b) Suppose now that H has the following general form:

$$H = -\vec{B} \cdot \hat{S} , \qquad \vec{B} \in \mathbb{R}^3 . \tag{6}$$

Let $\hat{S}(t)$ be the Heisenberg evolution of the spin operator with the Hamiltonian H. Prove that $\hat{S}(t)$ satisfies the following evolution equation:

$$\frac{d}{dt}\hat{S}(t) = \vec{B} \times \hat{S}(t) , \tag{7}$$

with $\times$ the usual vector product. This equation implies that the spin operator *precesses* at the frequency $|\vec{B}|$.
Hint. It might be useful to use the commutation relations of the Pauli matrices (check them):

$$[\sigma_a, \sigma_b] = 2i \sum_{c=x,y,z} \varepsilon_{abc}\sigma_c , \tag{8}$$

where ε_{abc} is the Levi-Civita symbol, defined as

$$\varepsilon_{abc} := \begin{cases} 1 & \text{if } (a,b,c) \text{ is an even permutation of } (x,y,z), \\ -1 & \text{if } (a,b,c) \text{ is an odd permutation of } (x,y,z), \\ 0 & \text{if } a = b, \text{ or } b = c, \text{ or } c = a . \end{cases} \tag{9}$$

Exercise 2. Let $n \in \mathbb{N}$ and $f \in L^1(\mathbb{R})$. Find conditions on the function f that imply the respective statements.

(a) $\hat{f} \in C^n(\mathbb{R})$.

(b) $\sup_{k\in\mathbb{R}} ||k|^n \hat{f}(k)| < \infty$.

(c) $\hat{f} \in L^1(\mathbb{R})$.

Exercise 3. Let $\hat{\psi} \in \mathcal{S}(\mathbb{R}^d)$, and consider the function $\hat{\psi}_t(k) := e^{i|k|^2 t}\hat{\psi}(k)$.

(a) Compute $\psi_t(x) = (\mathcal{F}^{-1}\hat{\psi}_t)(x)$.

Hint. Use that, by dominated convergence theorem:

$$\int_{\mathbb{R}^d} dk\, e^{ik\cdot x} e^{-i|k|^2 t}\hat{\psi}(k) = \lim_{R\to\infty} \int_{\mathbb{R}^d} dk\, e^{ik\cdot x} e^{-i|k|^2 t} e^{-\frac{|k|^2}{R^2}}\hat{\psi}(k)\ , \tag{10}$$

and recall how the product of functions behaves under Fourier transform.

(b) Show that $\psi_t \in L^\infty(\mathbb{R}^d)$ for all t, and compute $\lim_{t\to\infty} \psi_t(x)$.