Mathematical Quantum Theory
Exercise sheet 3
12.11.2019

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Exercise 1. Consider the free Schrödinger equation:

$$i\partial_t \psi_t = -\frac{1}{2}\Delta\psi_t \ , \tag{1}$$

with initial datum $\psi_0(x) = N^{-1}e^{-|x|^2}$ on $\mathbb{R}^d$, where $N = \pi^{\frac{d}{2}}$ is the normalization constant.

(i) Find the explicit solution of the equation, for all times.

(ii) Determine the expectation values of $\hat{x}$, $|\hat{x}|^2$, $\hat{p}$, $|\hat{p}|^2$ on ψ_t.

(iii) Let A be a bounded subset of $\mathbb{R}^d$. Compute:

$$\lim_{t\to\infty} \mathbb{P}_{\psi_t}(X(t) \in t^{\alpha}A) \ ; \tag{2}$$

discuss the cases $\alpha < 1$, $\alpha = 1$, $\alpha > 1$.

Exercise 2.

(i) Consider the free Schrödinger equation on a finite interval:

$$i\partial_t \psi_t = -\frac{1}{2}\Delta\psi_t \ , \qquad \psi_0 \in C^{\infty}((0;L)) \ , \tag{3}$$

with periodic boundary conditions, $\psi_t(0) = \psi_t(L)$. Prove or disprove the existence of stationary states, that is of solutions such that $\psi_t(x) = \psi(x)$ for all times (besides the trivial case $\psi(x) = 0$, of course).

(ii) Prove that all solutions of (3) with periodic boundary conditions are periodic in time: that is, there exists a time T such that $\psi_t = \psi_{t+T}$ for all t. How does T behave as a function of L? What happens with Dirichlet ($\psi_t(0) = \psi_t(L) = 0$) or with Neumann ($\partial_x\psi_t(0) = \partial_x\psi_t(L) = 0$) boundary conditions?

(iii) Consider now the heat equation:

$$\partial_t \psi_t = \frac{1}{2}\Delta\psi_t \ , \qquad \psi_0 \in C^{\infty}((0;L)) \ , \tag{4}$$

with $\psi_t(0) = \psi_t(L) = A$. Compute $\lim_{t\to\infty}\psi_t(x)$.

Exercise 3. Let H be a Hermitian matrix, and consider the Schrödinger equation on a finite dimensional Hilbert space, say $\mathbb{C}^n$:

$$i\partial_t \psi_t = H\psi_t \ , \qquad \psi_0 \in \mathbb{C}^n \ . \tag{5}$$

Prove that the dynamics is *recurrent* (or almost-periodic): for any $\epsilon > 0$ there exists $t_\epsilon > 0$ such that

$$\|\psi_{t_\epsilon} - \psi_0\| < \epsilon \ . \tag{6}$$

This is the quantum analogue of Poincaré recurrence theorem in classical mechanics.

Hint. Use that, by Dirichlet's approximation theorem, for any real numbers $\lambda_1, \ldots, \lambda_n$ and for any natural number K there exists integers $p_1, \ldots, p_n$, $1 \le q \le K$ such that

$$\left|\lambda_i - \frac{p_i}{q}\right| \le \frac{1}{qK^{1/n}} \ . \tag{7}$$