Mathematical Quantum Theory
Exercise sheet 4
19.11.2019

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Exercise 1. Let $\eta : \mathbb{R} \to \mathbb{R}$ be given by:

$$\eta(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \,. \end{cases} \tag{1}$$

(i) Prove that $T_\eta \in \mathcal{S}'(\mathbb{R})$.

(ii) Compute $\frac{d^n}{dx^n} T_\eta(f)$ for all $n \in \mathbb{N}$.

(iii) Prove that there exists no function $g_n \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ such that $\frac{d^n}{dx^n} T_\eta(f) = T_{g_n}$, for all $n \in \mathbb{N}$.

Exercise 2. Let $(c_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $|c_k| \leq Ck^p$ for some $p \in \mathbb{R}_+$. Let:

$$g_n(x) := \sum_{k=1}^{n} c_k \sin kx \,. \tag{2}$$

Prove that there exists $T \in \mathcal{S}'(\mathbb{R})$ such that T_{g_n} converges to T in the weak* sense.

Exercise 3. Prove the uniqueness of the solution of the free Schrödinger equation on $\mathcal{S}'(\mathbb{R}^d)$.

Exercise 4. Prove that $-\Delta$ is not a bounded operator on $L^2(\mathbb{R}^d)$.

Exercise 5. Let $\mathcal{H}$ be a Hilbert space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on $\mathcal{H}$, and let A be a bounded linear operator on $\mathcal{H}$. Prove that the weak convergence of A_n to A does not imply the strong convergence of A_n to A.

Hint. Take $\mathcal{H} = L^2(\mathbb{R}^d)$, and consider the bounded linear operator $Q : L^2 \to L^2$ such that $(Q_{A,B}\psi)(x) = \chi_B(x)\langle\chi_A, \psi\rangle$, with χ_A and χ_B the characteristic functions of two measurable, bounded subsets A and B of $\mathbb{R}^d$. Consider now the sequence $Q_n \equiv Q_{A,B_n}$, where $B_n = B + nz$ for a fixed $z \in \mathbb{R}^d$. For any $\varphi, \psi \in L^2(\mathbb{R}^d)$, compute $\langle\varphi, Q_n\psi\rangle$ and $\|Q_n\psi\|$ as $n \to \infty$.