Mathematical Quantum Theory
Exercise sheet 5
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Exercise 1.

(i) Let $\{\psi_n\}_{n\in\mathbb{N}_0}$ be the eigenstates of the one-dimensional harmonic oscillator, with Hamiltonian $H = -\frac{1}{2}\partial_x^2 + \frac{\omega^2}{2}x^2$, with $\omega > 0$. Let ΔX_{ψ_n}, ΔP_{ψ_n} be the variances of the position and of the momentum. Compute $\Delta X_{\psi_n}\Delta P_{\psi_n}$.

(ii) Prove that, if $|n-m| > k$ with $k \in \mathbb{N}_0$:

$$\langle \psi_n, \hat{x}^k \psi_m\rangle = \langle \psi_n, \hat{p}^k \psi_m\rangle = 0 \ , \tag{1}$$

with $\hat{p} = -i\partial_x$ the momentum operator.

Hint. Represent the operators $\hat{x}$, $\hat{p}$ in terms of the creation/annihilation operators:

$$A_\pm = \frac{1}{\sqrt{2}}\Big(\sqrt{\omega}x \pm \frac{1}{\sqrt{\omega}}\frac{d}{dx}\Big) \ . \tag{2}$$

Exercise 2. Let $H = -\frac{1}{2}\Delta + \frac{\omega}{2}|x|^2$ be the Hamiltonian of the three dimensional harmonic oscillator. The eigenstates of this Hamiltonian are given by $\prod_{i=1}^3 \psi_{n_i}(x_i)$, where $\{\psi_n\}_{n\in\mathbb{N}_0}$ are the eigenstates of the one-dimensional problem. Compute the eigenvalues of H, and determine their degeneracies:

$$\deg(E) := \text{number of independent eigenstates of } H \text{ with energy } E. \tag{3}$$

Exercise 3. Consider a one-dimensional quantum systems with wave function $\psi = \sum_{n=0}^N c_n\psi_n$, where N is finite and $\{\psi_n\}_{n\in\mathbb{N}_0}$ are the eigenstates of the one-dimensional harmonic oscillator with Hamiltonian $H = -\frac{1}{2}\partial_x^2 + \frac{\omega^2 x^2}{2}$, with $\omega > 0$.

(i) Compute $\langle \psi, x\psi\rangle$.

(ii) Let ψ_t be the solution of the time-dependent Schrödinger equation generated by H. Express $\langle \psi_t, x\psi_t\rangle$ in terms of $\langle \psi, x\psi\rangle$.

Exercise 4. Let $\ell^2(\mathbb{N})$ be the space of square-summable sequences, with scalar product $\langle \varphi, \psi\rangle = \sum_{n\in\mathbb{N}} \overline{\varphi_n}\psi_n$. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the (right) shift operator:

$$T\varphi := (0, \varphi_1, \varphi_2, \dots) \ . \tag{4}$$

Prove that $T \in \mathcal{L}(\ell^2(\mathbb{N}))$. Find the adjoint T^*.