Mathematical Quantum Theory
Exercise sheet 6
10.12.2019

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Exercise 1. Let $H \in \mathcal{L}(\mathcal{H})$, $H = H^*$. Prove that $(e^{-iHt})_{t\in\mathbb{R}}$ is a strongly continuous unitary group, generated by H. Then, prove the existence and uniqueness of the solution of the Schrödinger equation:

$$i\partial_t \psi_t = H\psi_t \,, \qquad \psi_0 \in \mathcal{H} \,. \tag{1}$$

Exercise 2. Consider one quantum particle confined in a box $[0; L]$. Consider the time-independent Schrödinger equation:

$$-\frac{1}{2}\Delta\psi = E\psi \,, \qquad \psi \in C^2((0; L)) \,, \qquad \psi(0) = \psi(L) = 0 \,. \tag{2}$$

(a) Find the solutions $(\psi_n)_{n\in\mathbb{N}}$ and the corresponding energies $(E_n)_{n\in\mathbb{N}}$
[Hint. Look for linear combinations of sine and cosine.]

(b) Compute the variances $\Delta\hat{x}_{\psi_n}$ and $\Delta\hat{p}_{\psi_n}$ for the n-th eigenstate ψ_n.

Exercise 3. Consider the one-dimensional quantum harmonic oscillator, with Hamiltonian $H = -\frac{1}{2}\Delta + \frac{1}{2}\omega^2 x^2$. Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ be the ground state. For any $\alpha \in \mathbb{C}$, define the *coherent state* as:

$$\psi(\alpha) = \exp\{\alpha A_+ - \overline{\alpha} A_-\}\psi_0 \,, \tag{3}$$

where $A_\pm$ are the creation/annihilation operators.

(a) Let A, B be linear operators on $\mathcal{S}(\mathbb{R})$, noncommuting in general. Suppose that $[A, B]$ commutes with A and B: $[[A, B], A] = [[A, B], B] = 0$. Then, it follows that:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \,. \tag{4}$$

Using (4), rewrite $\psi(\alpha)$ as a linear combination of eigenstates $(\psi_n)_{n\in\mathbb{N}}$ of the harmonic oscillator.

(b) Prove that $\Delta\hat{x}_{\psi(\alpha)}\Delta\hat{p}_{\psi(\alpha)}$ is minimal.
[Hint. Rewrite $\hat{x}$ and $\hat{p}$ in terms of the creation/annihilation operators.]

(c) Consider the time evolution $\psi_t = e^{-iHt}\psi(\alpha)$. Compute $\langle\psi_t, \hat{x}\psi_t\rangle$, $\langle\psi_t, \hat{p}\psi_t\rangle$, and compare with the solution of the equation of motion of the classical harmonic oscillator.

(d) Compute the energy $\langle\psi_t, H\psi_t\rangle$.

(e) Let $\mathbb{P}_{\psi_t}(P(t) \in A)$ be the probability that the momentum of the particle belongs to the set $A \in \mathbb{R}$:

$$\mathbb{P}_{\psi_t}(P(t) \in A) = \int_A dk\, |\hat{\psi}_t(k)|^2 \,. \tag{5}$$

Use the computation of item (d) to conclude that for any $\varepsilon > 0$ there exists $L_\varepsilon > 0$ large enough such that for all times $t \in \mathbb{R}$:

$$\mathbb{P}_{\psi_t}(P(t) \in [-L_\varepsilon, L_\varepsilon]) > 1 - \varepsilon \,. \tag{6}$$