Mathematical Quantum Theory
Exercise sheet 7
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Exercise 1. Let $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be a projection-valued measure. Let $\Omega, \Omega_1, \Omega_2$ be Borel sets. Prove the following properties.

(i) $P(\emptyset) = \mathbb{0}$, $P(\Omega^c) = \mathbb{1} - P(\Omega)$ (where $\mathbb{0}$ is the null operator and $\mathbb{1}$ is the identity on $\mathcal{H}$)

(ii) $P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2) - P(\Omega_1 \cap \Omega_2)$

(iii) $P(\Omega_1 \cap \Omega_2) = P(\Omega_1)P(\Omega_2)$

(iv) If $\Omega_1 \subseteq \Omega_2$, then $\langle \psi, P(\Omega_1)\psi\rangle \leq \langle \psi, P(\Omega_2)\psi\rangle$ for all $\psi \in \mathcal{H}$.

Exercise 2. Let $F(z)$ be a Herglotz function. Suppose that it can be written as:

$$F(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu(\lambda) \ , \tag{1}$$

for $z \in \mathbb{C}^+$ and where μ is a Borel measure. Prove that:

$$\mu((-\infty, \lambda]) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\lambda+\delta} \mathrm{Im} F(t + i\varepsilon) dt \ . \tag{2}$$

Rmk. This formula allows to recover the Borel measure from the function $F(z)$.

Hint. Recall:

$$\int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(\lambda - t)^2 + \varepsilon^2} d\lambda = \arctan\left(\frac{\lambda_2 - t}{\varepsilon}\right) - \arctan\left(\frac{\lambda_1 - t}{\varepsilon}\right) . \tag{3}$$

Then, use that (justify it):

$$\lim_{\varepsilon \to 0^+} \left(\frac{1}{\pi} \arctan\left(\frac{\lambda_2 - t}{\varepsilon}\right) - \arctan\left(\frac{\lambda_1 - t}{\varepsilon}\right)\right) = \frac{1}{2}\Big(\chi_{[\lambda_1, \lambda_2]}(t) + \chi_{(\lambda_1, \lambda_2)}(t)\Big) . \tag{4}$$

Exercise 3. Let T be a self-adjoint operator, with resolvent $R_z(T)$. Suppose that, for $\psi \in \mathcal{H}$:

$$\langle \psi, R_z(T)\psi\rangle = \int \frac{1}{\lambda - z} d\mu_\psi(\lambda) \ . \tag{5}$$

Suppose that the measure μ_ψ is supported on $(-\infty, a] \cup [b, c] \cup [d, \infty)$, with $a < b$ and $c < d$. Let $\Omega = [b, c]$, and prove that:

$$\mu_\psi(\Omega) = \frac{1}{2\pi i} \int_{\mathcal{C}_\Omega} dz \, \langle \psi, R_z(T)\psi\rangle \ , \tag{6}$$

where $\mathcal{C}_\Omega$ is a closed path in the complex plane enclosing Ω, which stays away from $(-\infty, a]$ and $[d, \infty)$.

Rmk. Denoting by P the projection-valued measure associated to $R_z(T)$ by the spectral theorem, using that $\langle \psi, P(\Omega)\psi\rangle = \mu_\psi(\Omega)$, Eq. (6) is equivalent to:

$$P(\Omega) = \frac{1}{2\pi i} \int_{\mathcal{C}_\Omega} dz \, R_z(T) \ . \tag{7}$$

Hint. Recall residue theorem in complex analysis.