Mathematical Quantum Theory
Exercise sheet 8
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Exercise 1. Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let A be a Hermitian matrix. By the spectral theorem:

$$\langle \psi, A\psi \rangle = \int d\mu_\psi(\lambda)\, \lambda \,. \tag{1}$$

Use Cauchy theorem for analytic functions to rewrite the measure μ_ψ in terms of the resolvent of A.

Exercise 2. Let $\mu_{\psi,\varphi}$ be the unique complex measure such that, for $z \in \mathbb{C} \setminus \mathbb{R}$:

$$\langle \varphi, R_z(H)\psi \rangle = \int d\mu_{\psi,\varphi}(\lambda) \frac{1}{\lambda - z} \,. \tag{2}$$

The existence of this measure has been proven in class, thanks to Herglotz theorem and using the polarization identity. The uniqueness is a consequence of Stieltjes formula (previous exercise sheet). Also, as proved in class, for any Borel set Ω we have $\mu_{\psi,\varphi}(\Omega) = \langle \psi, P(\Omega)\varphi \rangle$, for a suitable operator $0 \leq P(\Omega) \leq 1$. We also proved that $P(\Omega_1 \cap \Omega_2) = P(\Omega_1)P(\Omega_2)$. To prove that $P(\cdot)$ is a PVM, show that:

(i) $P(\mathbb{R})$ is the identity on $\mathcal{H}$,

(ii) P is strongly σ-additive.

Hint. (i) Suppose $P(\mathbb{R}) \neq 1$. Then, for any $\psi \in \mathcal{H}$, write $\psi = P(\mathbb{R})\psi + \varphi$ with $\varphi = (1 - P(\mathbb{R}))\psi$, which is $\neq 0$ by assumption. We will show that this gives rise to a contradiction. As proven in class, $P(\Omega)$ is a projection for any Borel set Ω; hence, $\varphi \in \ker P(\mathbb{R})$. Therefore, for any $\xi \in \mathcal{H}$, $d\mu_{\xi,P(\mathbb{R})\varphi} = 0$, which implies

$$\langle \xi, R_z(H)\varphi \rangle = 0 \,. \tag{3}$$

Here we used that, as we have seen in class, $d\mu_{\xi,P(\Omega)\varphi}(\lambda) = \chi_\Omega(\lambda) d\mu_{\xi,\varphi}(\lambda)$, for any Borel set Ω. Use Eq. (3), together with the fact that $R_z(H)$ is a bijection from $\mathcal{H}$ to $D(H)$ to prove that $\varphi = 0$.

(ii) To begin, prove that, for sequences of orthogonal projectors (P_n), weak convergence is equivalent to strong convergence:

$$\langle \psi, P_n\psi \rangle \to 0 \iff P_n\psi \to 0 \quad \textit{strongly.} \tag{4}$$

To prove (ii), let $\Omega = \bigcup_{j=1}^\infty \Omega_j$, with Ω_j disjoint. Take P_n to be $\sum_{j=1}^n P(\Omega_j)$ with Ω_j disjoint. To prove that P_n is an orthogonal projection, recall that $P(\Omega_j)P(\Omega_{j'}) = P(\Omega_j \cap \Omega_{j'})$. Then, prove that $P_n \to P(\Omega)$ weakly, by using the σ-additivity of Borel measures.

Exercise 3. Let A and B be two densely defined operators, such that $A \subset B$. Prove that $B^* \subset A^*$.

Hint. The condition $A \subset B$ is equivalent to $\Gamma(A) \subset \Gamma(B)$, with Γ the graph of the operator. Recall that $\Gamma(A^*) = (W\Gamma(A))^\perp$, with $W(\varphi_1, \varphi_2) = (-\varphi_2, \varphi_1)$. How does the inclusion relation behave after taking the orthogonal complement?

Rmk. Let $(H, D(H))$ be a self-adjoint operator, and let P be the PVM provided by the representation of its resolvent. Let Φ be the functional calculus generated by P. The above result is what we need to prove that if $H \subset \Phi(\lambda)$ (as we did in class), then $H = \Phi(\lambda)$.