Mathematical Quantum Theory
Exercise sheet 9
21.01.2020

Giovanna Marcelli

giovanna.marcelli@uni-tuebingen.de

Exercise 1. Let T be a finite rank operator on a Hilbert space $\mathcal{H}$:

$$T = \sum_{k=1}^{n} \alpha_k |\varphi_k\rangle\langle\psi_k| \ .$$

The families $(\varphi_k)_{k=1}^n$, $(\psi_k)_{k=1}^n$ are orthonormal and $\alpha_k \in \mathbb{C}$.

(i) Prove that T is bounded, uniformly in n.

(ii) Let T be a compact operator. Prove that T is bounded, using that K can be approximated in norm by finite rank operators.

Exercise 2. Let $(H, D(H))$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, inducing the decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{pp}$. Let $\psi \in \mathcal{H}_c$.

(i) Let K be a compact operator. Prove that:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|K e^{-iHt}\psi\| = 0 \ . \tag{1}$$

(ii) Let K be a relatively compact operator with respect to H. Prove that (1) still holds.

Hint. It is enough to prove the statement for a dense subspace of $\mathcal{H}_c$. Take $\psi \in \mathcal{H}_c \cap D(H)$. Use that, for $z \in \rho(H)$, the resolvent $R_z(H)$ establishes a bijection between $\mathcal{H}$ and $D(H)$, and recall the definition of relatively compact operator. Also, use that $\psi = P_c\psi$, where P_c is the orthogonal projection onto $\mathcal{H}_c$, and use that $[P_c, R_z(H)] = 0$.

(iii) Let K_n be a sequence of relatively compact operators with respect to H, such that $K_n \to 1_{\mathcal{H}}$ strongly. Let $\psi = \sum_{j=1}^N \alpha_j \psi_j$, with $\psi_j \in \mathcal{H}_{pp}$. Prove that:

$$\lim_{n\to\infty} \lim_{T\to\infty} \int_0^T dt\, \|K_n e^{-iHt}\psi\| \neq 0 \ .$$

Exercise 3. Consider a densely defined Schrödinger operator $H = -\Delta + V(x)$ on $L^2(\mathbb{R})$, with $V(x) \geq c|x|^2$ for some $c > 0$ and for all $x \in \mathbb{R}$. Suppose that $(H, D(H))$ is selfadjoint, on a dense domain $D(H)$. Use RAGE theorem to prove that $\sigma(H) = \sigma_{\mathrm{pp}}(H)$.

Hint. Use RAGE theorem with K_n replaced by χ_R, and that χ_R is relatively compact with respect to H.