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Foundations of QM: In-Class Problem Set 1

Problem 1: Verify that the Gaussian density

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

is normalized, has mean μ, and has standard deviation σ.

Problem 2: Find, for each of the following wave functions with given $a > 0$, the normalizing constant $\mathscr{N} > 0$ that will ensure $\|\psi\| = 1$.

(a) $\psi_1(x) = \mathscr{N}_1 \, e^{-x/a}$ with $x \in [0, \infty)$

(b) $\psi_2(\boldsymbol{x}) = \mathscr{N}_2 \, e^{-|\boldsymbol{x}|/a}$ with $\boldsymbol{x} \in \mathbb{R}^3$

(c) $\psi_3(\boldsymbol{x}) = \mathscr{N}_3 \, e^{-|\boldsymbol{x}|^2/a^2}$ with $\boldsymbol{x} \in \mathbb{R}^3$

Problem 3: Write $\psi(q) = R(q)e^{iS(q)/\hbar}$ with real-valued functions $R \geq 0$ and S. Show that

$$\boldsymbol{j}_i = \frac{1}{m_i} R^2 \, \nabla_i S \,. \tag{2}$$

Problem 4: An operator A is called *bounded* if there is a constant $M > 0$ such that

$$\|A\psi\| \leq M\|\psi\| \quad \forall \psi \,. \tag{3}$$

The smallest such constant is called the *operator norm* of A, written $\|A\|$. Justify the following statements:

(a) The Laplacian operator ∇^2 is unbounded.

(b) In $\mathbb{C}^n$, every operator is bounded.

(c) In $\mathbb{C}^n$, if A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\|A\| = \max_{j=1}^{n} |\lambda_j| \,. \tag{4}$$

(d) $\|A\| = \sup\limits_{\psi \neq 0} \dfrac{\|A\psi\|}{\|\psi\|} = \sup\limits_{\|\psi\|=1} \|A\psi\|$.

(e) The operator norm is a norm, i.e., $\|A\| > 0$ for $A \neq 0$, $\|zA\| = |z| \, \|A\|$, and $\|A + B\| \leq \|A\| + \|B\|$.

(f) Let $\mathscr{L}(\mathscr{H})$ be the space of bounded operators on the Hilbert space $\mathscr{H}$, equipped with the operator norm. One can show that $\mathscr{L}(\mathscr{H})$ is *complete*, i.e., every Cauchy sequence converges. Conclude that if A is bounded, then the exponential series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{5}$$

converges in operator norm.