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Foundations of QM: In-Class Problem Set 2

Problem 5: Dirac delta function
Let x be a 1-d variable and $g_\sigma(x)$ the Gaussian probability density,

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,. \tag{1}$$

The *Dirac δ function* can be defined heuristically as

$$\delta(x) = \lim_{\sigma\to 0} g_\sigma(x)\,. \tag{2}$$

Since $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$, the δ function is not a function in the ordinary sense; it is called a *distribution*. Based on the heuristic (2), one defines

$$\int_{\mathbb{R}} \delta(x-a)\, f(x)\, dx := \lim_{\sigma\to 0} \int_{\mathbb{R}} g_\sigma(x-a)\, f(x)\, dx\,. \tag{3}$$

It follows that if the function f is continuous at a, then

$$\int_{\mathbb{R}} \delta(x-a)\, f(x)\, dx = f(a)\,. \tag{4}$$

Mathematicians take this as the definition of the δ distribution; that is, they define $\delta(\cdot - a)$ as a linear operator from some function space such as $\mathscr{S}$ (Schwartz space) to $\mathbb{C}$, $f \mapsto f(a)$.

(a) Find the Fourier transform $\widehat{\delta_a}(k)$ of $\delta_a(x) = \delta(x-a)$ with arbitrary constant $a \in \mathbb{R}$. Find the function ψ whose Fourier transform is $\widehat{\psi}(k) = \delta(k-b)$ with arbitrary constant $b \in \mathbb{R}$.

(b) One defines the derivative δ' of the δ function by

$$\delta'(x) = \lim_{\sigma\to 0} g'_\sigma(x) \tag{5}$$

and its integrals by

$$\int_{\mathbb{R}} \delta'(x-a)\, f(x)\, dx := \lim_{\sigma\to 0} \int_{\mathbb{R}} g'_\sigma(x-a)\, f(x)\, dx\,. \tag{6}$$

Using integration by parts and (4), show that (for $f \in \mathscr{S}$)

$$\int_{\mathbb{R}} \delta'(x-a)\, f(x)\, dx = -f'(a)\,. \tag{7}$$

Problem 6: Delta function in higher dimension
(a) The d-dimensional Dirac delta function is defined by

$$\delta^d(\boldsymbol{x}-\boldsymbol{a}) = \delta(x_1-a_1)\cdots\delta(x_d-a_d) \tag{8}$$

Instead of $\delta^d(\boldsymbol{x}-\boldsymbol{a})$, one sometimes simply writes $\delta(\boldsymbol{x}-\boldsymbol{a})$. Verify that

$$\int_{\mathbb{R}^d} \delta^d(\boldsymbol{x}-\boldsymbol{a})\, f(\boldsymbol{x})\, d^d\boldsymbol{x} = f(\boldsymbol{a})\,. \tag{9}$$

(b) For a generalized orthonormal basis (GONB) with continuous parameter, $\{\phi_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{R}^d\}$, one requires that

$$\langle \phi_{\boldsymbol{k}_1} | \phi_{\boldsymbol{k}_2} \rangle = \delta^d(\boldsymbol{k}_1 - \boldsymbol{k}_2)\,. \tag{10}$$

Verify this relation for the basis functions of Fourier transformation,

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) = (2\pi)^{-d/2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{11}$$

(c) Verify that every $\phi_{\boldsymbol{k}}$ as given by (11) is an eigenfunction of each momentum operator $P_j = -i\hbar\partial/\partial x_j$, $j = 1, \ldots, d$. Thus, (11) defines a GONB that simultaneously diagonalizes all P_j. It is therefore called the *momentum basis*.

(d) Now consider another basis, given by

$$\phi_{\boldsymbol{y}}(\boldsymbol{x}) = \delta^d(\boldsymbol{x}-\boldsymbol{y})\,. \tag{12}$$

Verify Eq. (10) for these functions. Then verify that every $\phi_{\boldsymbol{y}}$ is an eigenfunction of each position operator $X_j\psi(\boldsymbol{x}) = x_j\psi(\boldsymbol{x})$, $j = 1, \ldots, d$. Thus, (12) defines a GONB that simultaneously diagonalizes all X_j. It is therefore called the *position basis*.

Problem 7: Here is a generalization of the spectral theorem, for simplicity formulated in finite dimension: If the self-adjoint $d \times d$ matrices A and B commute, then they can be simultaneously unitarily diagonalized, i.e., there is an orthonormal basis $\{\phi_1, \ldots, \phi_d\}$ such that each ϕ_j is an eigenvector of A and an eigenvector of B.

Show that a $d \times d$ matrix C can be unitarily diagonalized iff C commutes with $C^\dagger$. Such a matrix is called "normal."