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Foundations of Quantum Mechanics: Assignment 6

Exercise 22: Essay question
Explain why the interference pattern in the double-slit experiment disappears if a detector measures through which slit the electron went, even if we ignore the outcome of the detection. Derive this fact from the projection postulate, using the observable (10.16) as a model of a detector, where B is a suitable neighborhood of one slit. Explain why the pattern on the screen must be the same as if one slit were closed in each run, each slit in 50% of the runs. (Use formulas where appropriate.)

Exercise 23: Projections
(a) Show that a self-adjoint operator P with $P^2 = P$ (i.e., a projection) has no other eigenvalues than 0 and 1.

(b) Show that every self-adjoint operator P with eigenvalues 0 and 1 satisfies $P^2 = P$ (and thus is a projection).

(c) Suppose that $P : \mathscr{H} \to \mathscr{H}$ is a projection with range $\mathscr{K}$; one says that P is the projection to $\mathscr{K}$. Show that $I - P$ is the projection to the orthogonal complement of $\mathscr{K}$, i.e., to
$\mathscr{K}^\perp = \{\phi \in \mathscr{H} : \langle\phi|\psi\rangle = 0 \, \forall \psi \in \mathscr{K}\}$.

(d) Suppose that P is the projection to $\mathscr{K}$. Show that the element in $\mathscr{K}$ closest to a given vector $\psi \in \mathscr{H}$ is $P\psi$.

Exercise 24: Iterated Stern-Gerlach experiment
Consider the following experiment on a single electron. Suppose it has a wave function of the product form $\psi_s(\boldsymbol{x}) = \phi_s \, \chi(\boldsymbol{x})$, and we focus only on the spinor. The initial spinor is $\phi = (1, 0)$.

(a) A Stern–Gerlach experiment in the y-direction (or σ_2-measurement) is carried out, then a Stern–Gerlach experiment in the z-direction (or σ_3-measurement). Both measurements taken together have four possible outcomes: up-up, up-down, down-up, down-down. Find the probabilities of the four outcomes.

(b) As in (a), but now the z-experiment comes first and the y-experiment afterwards. (The fact that the answer to (b) is different from that to (a) is often expressed by saying that the observables σ_2 and σ_3 "cannot be measured simultaneously.")

(c) More generally, if the two self-adjoint matrices A and B have spectral decomposition $A = \sum_\alpha \alpha P_\alpha$ and $B = \sum_\beta \beta Q_\beta$, what is the joint distribution of the outcomes if A is measured first and B right afterwards? What if B is measured first and A right afterwards? Show that the two agree iff every P_α commutes with every Q_β. (This happens iff $AB = BA$.)

Please turn over.

Exercise 25: Lie algebras of $SO(3)$ and $SU(2)$
A *Lie group* G, named after Sophus Lie (1842–1899), is a group that is also a manifold such that the group multiplication and inversion are smooth mappings. Examples of Lie groups include $GL(n), SO(n), U(n), SU(n)$. The elements infinitesimally close to 1 in G form the *Lie algebra* g of G; more precisely, g is the tangent space of 1, which is here the set

$$\left\{ \frac{dA}{dt}(t=0) \middle| A : (-1,1) \to G \text{ smooth}, A(0) = 1 \right\}.$$

(a) Determine the Lie algebras $so(3)$ and $su(2)$ as subspaces of the space of all real 3×3 (complex 2×2) matrices.

(b) The *exponential mapping* $\exp : g \to G$ can be heuristically understood as follows: For $X \in g$, a corresponding group element infinitesimally close to 1 can be written as $1 + X/n$ with n a large natural number (so $1/n$ serves as an infinitesimal dt). Hence, roughly speaking, $(1 + X/n) \in G$, hence $(1 + X/n)^n \in G$; take the limit $n \to \infty$ to obtain $\exp(X) =: e^X$. Verify that the matrix exponential (defined by the exponential series) actually maps $so(3)$ to $SO(3)$ and $su(2)$ to $SU(2)$. (Hint: diagonalize $X \in g$.)

(c) For $X, Y \in g$, what does group multiplication of e^X and e^Y look like? We know that the solution Z of $e^Z = e^X e^Y$ is $Z = X + Y$ if X and Y commute, but not in general. A version of the *Baker–Campbell–Hausdorff formula* says that

$$\text{the solution of } e^Z = e^{-tX} e^{-tY} e^{t(X+Y)} \text{ is } Z = \tfrac{1}{2} t^2 [X, Y] + O(t^3)$$

as $t \to 0$, with $[X, Y] = XY - YX$ the *commutator* or *Lie bracket*. The Lie bracket is an operation on g that encodes how the group multiplication deviates from addition in g. Thus, one defines a *Lie algebra* in general as a vector space together with a bracket $[\cdot, \cdot] : g \times g \to g$ that is anti-symmetric, bilinear, and satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\,.$$

Verify that $so(3)$ and $su(2)$ (with commutators as Lie brackets) are isomorphic Lie algebras. (Hint: The Pauli matrices have something to do with rotations about the x-, y-, and z-axis.)

Hand in: Tuesday November 26, 2019, in class

Reading assignment due Friday November 29, 2019: A. Einstein, *Reply to Criticisms*, pages 665–688 in P. Schilpp (editor): *Albert Einstein, Philosopher–Scientist* (1949). Read pages 665–672 and the first quarter of 673.