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Foundations of Quantum Mechanics: Assignment 11

Exercise 43: Essay question. Any proof of nonlocality must involve an *entangled* wave function of *at least two* particles. Explain.

Exercise 44: Positive operators
An operator $S : \mathbb{C}^d \to \mathbb{C}^d$ is positive iff $\langle\psi|S\psi\rangle \geq 0$ for all ψ. Are the following statements about operators on $\mathbb{C}^d$ true or false? Justify your answers.

(a) $R^\dagger R$ is always a positive operator.

(b) If E is a positive operator, then so is $R^\dagger E R$.

(c) The positive operators form a subspace of the space of self-adjoint operators.

(d) The sum of two projections is positive only if they commute.

(e) e^{At} is a positive operator for every self-adjoint A and $t \in \mathbb{R}$.

Exercise 45: Consecutive quantum measurements
Let $A_1, \ldots, A_n$ be self-adjoint operators in $\mathscr{H}$ whose spectra $\sigma(A_k)$ are purely discrete (i.e., countable), so that

$$A_k = \sum_{\alpha \in \sigma(A_k)} \alpha \, P_{k,\alpha}$$

with $P_{k,\alpha}$ the projection to the eigenspace of A_k with eigenvalue α. Consider a quantum system with initial wave function $\psi_0 \in \mathscr{H}$ with $\|\psi_0\| = 1$ at time t_0. At times $t_1 < t_2 < \ldots < t_n$, ideal quantum measurements of $A_1, \ldots, A_n$ (respectively) are carried out with outcomes $Z_1, \ldots, Z_n \in \mathbb{R}$ ($t_0 < t_1$). Show that there is a POVM E on $\mathbb{R}^n$ such that

$$\mathbb{P}\Big((Z_1, \ldots, Z_n) \in B\Big) = \langle\psi_0|E(B)|\psi_0\rangle$$

and give an explicit expression for $E(B)$.

Exercise 46: POVMs
(a) Suppose E_1 and E_2 are POVMs on $\mathscr{Z}_1$ and $\mathscr{Z}_2$, respectively, both acting on $\mathscr{H}$; let $q_1, q_2 \in [0,1]$ with $q_1 + q_2 = 1$. Show that $E(B) := q_1 E_1(B \cap \mathscr{Z}_1) + q_2 E_2(B \cap \mathscr{Z}_2)$ defines a POVM on $\mathscr{Z}_1 \cup \mathscr{Z}_2$.

(b) Suppose experiment $\mathscr{E}_1$ has distribution of outcomes $\langle\psi|E_1(\cdot)|\psi\rangle$, and $\mathscr{E}_2$ has distribution of outcomes $\langle\psi|E_2(\cdot)|\psi\rangle$. Describe an experiment with distribution of outcomes $\langle\psi|E(\cdot)|\psi\rangle$.

(c) Give an example of a POVM for which the E_z do not pairwise commute. *Suggestion*: Choose $E_1(z)$ that does not commute with $E_2(z')$ for $\mathscr{Z}_1 \cap \mathscr{Z}_2 = \emptyset$.

Hand in: Tuesday January 14, 2020, in class

Reading assignment due Friday January 17, 2020:
J. Bell: Bertlmann's Socks and the Nature of Reality. *Journal de Physique* **42**: C2 41–61 (1981)