Notation $\partial_i f = \dfrac{\partial f}{\partial x_i}$

4.15 Satz von Schwarz Sei $G \subset \mathbb{R}^n$ offen, W $\mathbb{K}$-VR, $\dim W < \infty$, $f \in C^2(G, W)$. Dann

$$\partial_i \partial_j f(x) = \partial_j \partial_i f(x) \quad \forall x \in G$$
$$\forall i, j \in \{1 \dots n\}$$

Bsp $h(x,y)$ aus ÜA 18.

$$\frac{\partial^2 h}{\partial y \partial x}(0,0) = -1 \neq 1 = \frac{\partial^2 h}{\partial x \partial y}(0,0)$$

Bsp $f \in C^5(\mathbb{R}^n, \mathbb{R})$, $\partial_3 \partial_1 \partial_4 \partial_3 \partial_1 f$
$$= \partial_1^2 \partial_3^2 \partial_4 f.$$

Def 4.16 $f \in C^2(G, \mathbb{R})$, $x \in G$

$$\operatorname{Hess} f(x) = \begin{bmatrix} \partial_1 \partial_1 f(x) & \partial_1 \partial_2 f(x) \cdots & \partial_1 \partial_n f(x) \\ \vdots & & \\ \partial_n \partial_1 f(x) & \cdots & \partial_n \partial_n f(x) \end{bmatrix}$$

heißt Hesse-Matrix von f in x.

Sie ist symm.

Bsp 4.17 $\Delta f(x) = \operatorname{Spur}(\operatorname{Hess} f(x))$
$$= \sum_{i=1}^n \partial_i^2 f.$$

Bew des Satzes von Schwarz

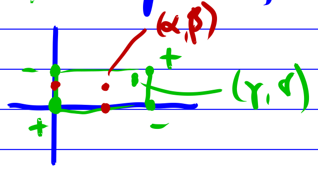
$\text{Satz}(W) \Leftarrow \text{Satz}(\mathbb{R}^m) \Leftarrow \text{Satz}(\mathbb{R})$

also oBdA $W = \mathbb{R}$.

OBdA $n=2$, $i=1$, $j=2$, $x=0$.

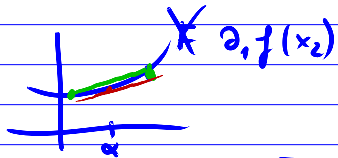
Zu zeigen: $\partial_1 \partial_2 f(0) = \partial_2 \partial_1 f(0)$.

$$y := \underbrace{f(x_1,x_2) - f(x_1,0)}_{=:F(x_1)} - f(0,x_2) + f(0,0)$$



$$y = F(x_1) - F(0)$$

$$\overset{\text{MWS}}{=} F'(\alpha)\, x_1 \qquad [\alpha = \alpha(x_1,x_2),\ |\alpha| \le |x_1|]$$



$$= \big(\partial_1 f(\alpha,x_2) - \partial_1 f(\alpha,0)\big)\, x_1$$

$$\overset{\text{MWS}}{=} \partial_2 \partial_1 f(\alpha,\beta)\, x_1 x_2 \qquad [\beta = \beta(x_1,x_2),\ |\beta| \le |x_2|]$$

Andererseits

$$y = \underbrace{f(x_1,x_2) - f(0,x_2)}_{=:G(x_2)} \underbrace{- f(x_1,0) + f(0,0)}_{= -G(0)}$$

$y = G(x_2) - G(0)$

$$\overset{MWS}{=} G'(\gamma)\, x_2$$

$$= \big(\partial_2 f(x_1, \gamma) - \partial_2 f(0, \gamma)\big)\, x_2$$

$$\overset{MWS}{=} \big(\partial_1 \partial_2 f(\delta, \gamma) \big)\, x_1 x_2$$

mit $|\gamma| \le |x_1|$, $|\delta| \le |x_2|$.

Also: für $x_1 x_2 \neq 0 \Rightarrow$

$$\partial_2 \partial_1 f(\alpha, \beta) = \partial_1 \partial_2 f(\gamma, \delta)$$

Vor: $\partial_2 \partial_1 f$ st. $\Rightarrow$ $x_1, x_2 \to 0$: $\alpha \to 0$, $\beta \to 0$; $x_1, x_2 \to 0$: $\gamma \to 0$, $\delta \to 0$

$\partial_1 \partial_2 f$ st. $\Rightarrow$

$$\partial_2 \partial_1 f(0,0) = \partial_1 \partial_2 f(0,0)$$

$\square$

Def 4.18 Sei $G \subset \mathbb{R}^3$ offen,

$v : G \to \mathbb{R}^3$ VF, part. diffbar

$$\operatorname{rot} v = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} = \nabla \times v$$

Rotation von v

engl. curl

$$= \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Korollar 4.20 Sei $G \subset \mathbb{R}^3$ offen

a) Für $f \in C^2(G)$ gilt $\operatorname{rot}(\operatorname{grad} f) = 0$

b) Für $v \in C^2(G, \mathbb{R}^3)$ gilt $\operatorname{div}(\operatorname{rot} v) = 0$.

Bew z.B. 1. Komp. von a)

$$\operatorname{rot}_1(\operatorname{grad} f) = \partial_2 \partial_3 f - \partial_3 \partial_2 f = 0. \qquad \square$$