Kap. 5: Taylor-Formel und lok. Extrema

1d: $g: \mathbb{R} \to \mathbb{R}, \quad x, h \in \mathbb{R}$

$$g(x+h) = g(x) + g'(x)\,h + \frac{1}{2} g''(x)\,h^2 + \dots$$

$$\dots + \frac{1}{k!} g^{(k)}(x)\,h^k + o\left(|h|^k\right)$$

nd: $f: \mathbb{R}^n \longrightarrow \mathbb{R}, \ \underline{x}, \underline{h} \in \mathbb{R}^n, \ \underline{h} = t\underline{v}, \ \|\underline{v}\| = 1$

$t = \|\underline{h}\|$

$$f(\underline{x}+\underline{h}) = f(\underline{x} + t\underline{v}) =: g(t) =$$

$$= f(\underline{x}) + \frac{\partial f}{\partial \underline{v}}(\underline{x})\,t + \frac{1}{2}\frac{\partial^2 f}{\partial \underline{v}^2}(\underline{x})\,t^2 + \dots$$

$$\dots + \frac{1}{k!}\frac{\partial^k f}{\partial \underline{v}^k}(\underline{x})\,t^k + o\left(|t|^k\right)$$

$$= \underbrace{\sum_{m=0}^{k} \frac{1}{m!} (\underline{h}\cdot\nabla)^m f(\underline{x})}_{\text{Taylor-Polynom in } \underline{h}} + o\left(\|\underline{h}\|^k\right)$$

$$= \sum_{m=0}^{k} \frac{1}{m!} D^m f(\underline{x})\,(\underbrace{\underline{h}, \underline{h}, \dots, \underline{h}}_{m}) + o\left(\|\underline{h}\|^k\right)$$

$Df(\underline{x}): \ \mathbb{R}^n \longrightarrow \mathbb{R}$

$D^2 f(\underline{x}): \ \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ bilinear

$D^3 f(\underline{x}): \ \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$\vdots$

$D^k f(\underline{x}): (\mathbb{R}^n)^k \longrightarrow \mathbb{R}$

multilinear

$$(\underline{h}\cdot\nabla)^m = \left(\sum_{j=1}^{n} h_j \partial_j\right)^m$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} h_{j_1} h_{j_2} \cdots h_{j_m} \partial_{j_1} \cdots \partial_{j_m}$$

$$\frac{\partial^2 f}{\partial \underline{v}^2}(\underline{x}) t^2 = (\underline{h}\cdot\nabla)^2 f(\underline{x})$$

$$= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} h_{j_1} h_{j_2} \partial_{j_1} \partial_{j_2} f(\underline{x})$$

$$= \langle \underline{h}, \ \mathrm{Hess} f(\underline{x}) \ \underline{h} \rangle.$$

Also $f(\underline{x}+\underline{h}) = \sum_{m=0}^{k} \frac{1}{m!} \sum_{j_1,\dots,j_m=1}^{n} h_{j_1} \cdots h_{j_m} \partial_{j_1} \cdots \partial_{j_m} f(\underline{x})$

$+ o(\|\underline{h}\|^k)$

Satz von Schwarz $\Rightarrow$ Permutationen von $j_1 \dots j_m$ liefern denselben Beitrag

<u>Notation</u> $\alpha_j := \#\{ r \in \{1\dots m\} \mid j_r = j \} \in \mathbb{N}_0$

$\underline{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$

$\underline{h}^{\underline{\alpha}} := h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}$

$\partial^{\underline{\alpha}} f := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f$

$\underline{\alpha}! := \alpha_1! \, \alpha_2! \cdots \alpha_n!$

$|\underline{\alpha}| := \sum_{j=1}^{n} \alpha_j$, $\quad \partial^{\underline{\alpha}} f = \dfrac{\partial^{|\underline{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$

Multinomialkoeffizient

$$\binom{k}{\alpha_1\ \alpha_2 \cdots \alpha_n} = \frac{k!}{\alpha_1!\,\alpha_2! \cdots \alpha_n!} = \frac{k!}{\underline{\alpha}!}$$

für $|\underline{\alpha}| = k$

$$\Rightarrow \sum_{j_1 \cdots j_m = 1}^{n} h_{j_1} \cdots h_{j_m} \partial_{j_1} \cdots \partial_{j_m} = \sum_{|\underline{\alpha}| = m} \frac{k!}{\underline{\alpha}!} \underline{h}^{\underline{\alpha}} \partial^{\underline{\alpha}}$$

Also $f(\underline{x} + \underline{h}) = \sum_{|\underline{\alpha}| \le k} \frac{1}{\underline{\alpha}!} \underline{h}^{\underline{\alpha}} \partial^{\underline{\alpha}} f(\underline{x}) + o(\|\underline{h}\|^k)$

Rigoros: $\quad G \subset \mathbb{R}^n$ offen

<u>Lemma 5.2</u> $f \in C^k(G, \mathbb{R}) \Rightarrow$

$$\frac{\partial^k f}{\partial \underline{h}^k} = (\underline{h} \cdot \nabla)^k f = \sum_{|\underline{\alpha}| = k} \frac{k!}{\underline{\alpha}!} \underline{h}^{\underline{\alpha}} \partial^{\underline{\alpha}} f.$$