5.4 Satz von Taylor

Sei $f \in C^{k+1}(G, \mathbb{R})$, $\underline{x} \in G \subset \mathbb{R}^n$ offen, $\underline{h} \in \mathbb{R}^n$, $\{\underline{x} + t\underline{h} \mid t \in [0,1]\} \subset G$. Dann $\exists\, \theta \in [0,1]$

$$f(\underline{x}+\underline{h}) = \underbrace{\sum_{|\underline{\alpha}| \leq k} \frac{1}{\underline{\alpha}!} \underline{h}^{\underline{\alpha}} \partial^{\underline{\alpha}} f(\underline{x})}_{=: P^{(k)}_{f,\underline{x}}(\underline{h})} + \underbrace{\sum_{|\underline{\alpha}| = k+1} \frac{1}{\underline{\alpha}!} \underline{h}^{\underline{\alpha}} \partial^{\underline{\alpha}} f(\underline{x}+\theta\underline{h})}_{\text{Restglied}}$$

Bew Setze $\underline{\gamma}(t) = \underline{x} + t\underline{h}$

$$\varphi(t) = f(\underline{\gamma}(t)), \quad \varphi \in C^{k+1}([0,1], \mathbb{R})$$

Satz von Taylor in $\mathbb{R}^1 \Rightarrow \exists\, \theta \in [0,1]$:

$$\varphi(1) = \sum_{m=0}^{k} \frac{\varphi^{(m)}(0)}{m!} + \frac{\varphi^{(k+1)}(\theta)}{(k+1)!} \Rightarrow \text{Beh.}$$

$\square$

Korollar 5.6 Sei $f \in C^k(G, \mathbb{R})$, $\underline{x} \in G \subset \mathbb{R}^n$ offen. Dann

$$f(\underline{x}+\underline{h}) = \sum_{|\underline{\alpha}| \leq k} \frac{1}{\underline{\alpha}!} \underline{h}^{\underline{\alpha}} \partial^{\underline{\alpha}} f(\underline{x}) + \underbrace{o\left(\|\underline{h}\|^k\right)}_{\varphi}.$$

Bew G offen $\Rightarrow \exists\, \delta > 0$: $\underbrace{B_\delta(\underline{x})}_{\text{konvex}} \subset G$

Satz von Taylor 5.4 $\Rightarrow$ $\forall \underline{h} \in B_\delta(\underline{0})$
$\exists\, \theta(\underline{h}) \in [0,1]$:

$$f(\underline{x}+\underline{h}) = \sum_{|\alpha| \le k-1} \frac{\underline{h}^\alpha}{\alpha!} \partial^\alpha f(\underline{x}) + \sum_{|\alpha|=k} \frac{\underline{h}^\alpha}{\alpha!} \partial^\alpha f(\underline{x}+\theta \underline{h}).$$

Setze $\varphi : B_\delta(\underline{0}) \to \mathbb{R}$ als

$$\varphi(\underline{h}) = \sum_{|\alpha|=k} \frac{\underline{h}^\alpha}{\alpha!} \left(\partial^\alpha f(\underline{x}+\theta(\underline{h})\underline{h}) - \partial^\alpha f(\underline{x}) \right)$$

Zu zeigen: $\varphi(\underline{h}) = o(\|\underline{h}\|^k) \iff$

$$\frac{\varphi(\underline{h})}{\|\underline{h}\|^k} \xrightarrow{\underline{h}\to 0} 0$$

Tatsächl. $$\frac{\varphi(\underline{h})}{\|\underline{h}\|^k} = \sum_{|\alpha|=k} \frac{1}{\alpha!} \frac{\underline{h}^\alpha}{\|\underline{h}\|^k} \underbrace{\left(\partial^\alpha f(\underline{x}+\theta(\underline{h})\underline{h}) - \partial^\alpha f(\underline{x}) \right)}_{\xrightarrow{\underline{h}\to \underline{0}} 0}$$

weil $\partial^\alpha f$ st.

$$= \frac{h_1^{\alpha_1}}{\|\underline{h}\|^{\alpha_1}} \frac{h_2^{\alpha_2}}{\|\underline{h}\|^{\alpha_2}} \cdots \frac{h_n^{\alpha_n}}{\|\underline{h}\|^{\alpha_n}} \in [-1, 1]$$

$\square$

Bem 5.7

$\underline{k=0}$: $f(\underline{x}+\underline{h}) = f(\underline{x}) + o(1)$ ($\iff f$ st.)

$\underline{k=1}$: $f(\underline{x}+\underline{h}) = f(\underline{x}) + Df(\underline{x})\underline{h} + o(\|\underline{h}\|)$
($\iff f$ diffbar)

$\underline{k=2}$: $f(\underline{x}+\underline{h}) = f(\underline{x}) + Df(\underline{x})\,\underline{h} +$

$+ \frac{1}{2}\langle \underline{h}, \operatorname{Hess} f(\underline{x})\,\underline{h}\rangle + o(\|\underline{h}\|^2)$