Beweis des Satzes von Fubini

Satz: $f:[a,b]\times[c,d]\to\mathbb{R}$ st. $\Rightarrow$

$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

Vorbereitung

Lemma 10.2 Sei $U\subset\mathbb{R}^n$, $f:[a,b]\times U\to\mathbb{R}$ st.

$U\ni y_k\to y_*\in U$. Die Fktfolge

$$f_k(x)=f(x,y_k)$$

konv. glm. gegen $f_*(x)=f(x,y_*)$.

Lemma 10.3 Sei $f:[a,b]\times U\to\mathbb{R}$ st. ($U\subset\mathbb{R}^n$)

Dann ist $g:U\to\mathbb{R}$, $g(y)=\int_a^b f(x,y)\,dx$ stetig.

Bew Allg. gilt: Wenn $f_k\to f_*$ glm., dann $\int_a^b f_k(x)\,dx \xrightarrow{k\to\infty} \int_a^b f_*(x)\,dx$.

Hier sei $U\ni y_k\to y_*\in U$, also nach 10.2

$$g(y_k)=\int_a^b f(x,y_k)\,dx \xrightarrow{k\to\infty} \int_a^b f(x,y_*)\,dx = g(y_*)$$

$\square$

Lemma 10.4 Sei $U \subset \mathbb{R}^n$ offen,

$f : [a,b] \times U \to \mathbb{R}$ st.

$\exists \dfrac{\partial f}{\partial y_j}(\underset{[a,b]}{x}, \underset{U}{y})$ st. $\forall j \in \{1,\dots,n\}$. Dann ist

$g : U \to \mathbb{R}$, $g(y) = \int_a^b f(x,y)\,dx$

C^1 und $\dfrac{\partial g}{\partial y_j} = \int_a^b \dfrac{\partial f}{\partial y_j}(x,y)\,dx$.

" $\dfrac{\partial}{\partial y_j} \int_a^b = \int_a^b \dfrac{\partial}{\partial y_j}$ "

Bew Sei $y \in U$ fest.

Brauchen: $\dfrac{f(x, y + h_k e_j) - f(x,y)}{h_k} \xrightarrow{k\to\infty} \dfrac{\partial f}{\partial y_j}(x,y)$

für jede Nullfolge h_k glm. in x

denn dann $\dfrac{g(y + h_k e_j) - g(y)}{h_k}$

$$= \int_a^b \frac{f(x,y+h_k e_j) - f(x,y)}{h_k}\,dx$$

$$\xrightarrow{k\to\infty} \int_a^b \frac{\partial f}{\partial y_j}(x,y)\,dx$$

Kriegen: Mittelwertsatz

$$\frac{f(x,y+h_k e_j)-f(x,y)}{h_k} = \underbrace{\frac{\partial f}{\partial y_j}\big(x, y+\overset{[0,1]}{\overset{\downarrow}{\theta}}(x,h_k)\, h_k e_j\big)}$$

$$\frac{\partial f}{\partial y_j} \text{ st.} \overset{\text{wie Beweis 10.2}}{\Longrightarrow} \text{glm. konv. in } x \text{ für } k\to\infty. \quad \square$$

10.5 Idee

10.4 sagt

$$\begin{array}{ccc} \cdot & \xrightarrow{\int dx} & \cdot \\ {\scriptstyle \partial_{y_j}}\downarrow & \% & \downarrow{\scriptstyle \partial_{y_j}} \\ \cdot & \xrightarrow[\int dx]{} & \cdot \end{array}$$

Brauchen:

$$\underbrace{\int_a^b dx \int_c^d dy\, f(x,y)} = \left| \begin{array}{ccc} \cdot & \xrightarrow{\int dx} & \cdot \\ {\scriptstyle \int dy}\uparrow & \% & \uparrow{\scriptstyle \int dy} \\ \cdot & \xrightarrow[\int dx]{} & \cdot \end{array}\right.$$

$$\underbrace{\int_c^d dt\, \frac{\partial}{\partial t}} \int_a^b dx \int_c^t dy\, f(x,y) \overset{10.4}{=}$$

$$[\ ]_c^d \quad \text{HS} \left| \int_c^d dt \int_a^b dx\, \underbrace{\frac{\partial}{\partial t} \int_c^t dy\, f(x,y)}_{f(x,t) \text{ HS}} = \int_c^d dy \int_a^b dx\, f(x,y) \right.$$

Vor HS: st.

Bew Satz 10.5: Sei $g(x) = \int_c^d f(x,y)\,dy$

$g : [a,b] \to \mathbb{R}$

$h(y) = \int_a^b f(x,y)\,dx$

$h : [c,d] \to \mathbb{R}$

Lemma 10.3 $\Rightarrow$ g, h st.

$$\tilde{g}(x,t) := \int_c^t f(x,y)\,dy$$

$\tilde{g} : [a,b] \times [c,d] \to \mathbb{R}$ ist st.,

denn $|\tilde{g}(x_1,t_1) - \tilde{g}(x_2,t_2)|$

$$\leq \underbrace{|\tilde{g}(x_1,t_1) - \tilde{g}(x_1,t_2)|} + \underbrace{|\tilde{g}(x_1,t_2) - \tilde{g}(x_2,t_2)|}$$

$$= \left| \int_{t_2}^{t_1} f(x_1,y)\,dy \right| \qquad \left| \int_c^{t_2} f(x_1,y) - f(x_2,y) \right|$$

$$\leq |t_1 - t_2| \underbrace{\sup|f|}_{<\infty} \xrightarrow{t_1 \to t_2} 0 \qquad \leq \int_c^{t_2} dy\, |f(x_1,y) - f(x_2,y)| \xrightarrow{x_1 \to x_2} 0$$

Lemma 10.3.

$$\text{HS} \Rightarrow \exists \frac{\partial \tilde{g}}{\partial t} = f(x,t) \text{ st.} \overset{\text{Lemma 10.4}}{\Longrightarrow}$$

$$t \mapsto \phi(t) := \int_a^b \tilde{g}(x,t)\,dx \quad \text{ist } C^1([c,d], \mathbb{R})$$

$$\text{und } \phi'(t) = \int_a^b \frac{\partial \tilde{g}}{\partial t}(x,t)\,dx = \int_a^b f(x,t)\,dx$$

Also $\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_c^d \phi'(t)\,dt$

$$\overset{HS}{=} \phi(d) - \underbrace{\phi(c)}_{0 \text{ weil } \tilde{g}(x,c)=0} = \phi(d) = \int_a^b dx \left(\int_a^d dy\, f(x,y) \right)$$

$\square$