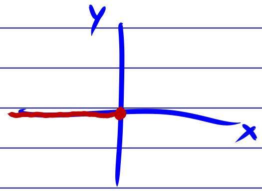
Integral in Polarkoordinaten

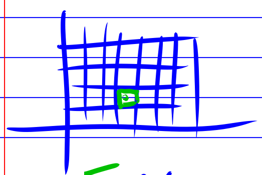
Polarkoo:

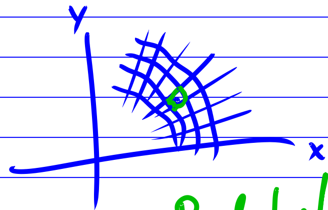
$$\Phi : (r,\varphi) \longmapsto \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$



$$\Phi : (0,\infty) \times (-\pi,\pi) \longrightarrow \mathbb{R}^2 \setminus \{(x,y) : x \le 0\}$$

$$\tilde f(r,\varphi) = f \circ \Phi, \qquad \int_{\mathbb{R}^2} f(x,y)\, d(x,y)$$

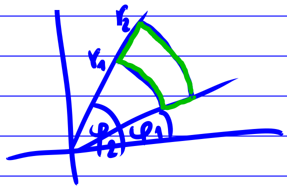


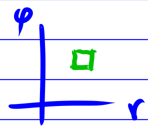


$$\int f\, dA \approx \sum_{ij} f(x_i, y_j)\, \underbrace{dA_{ij}}_{dx_i\, dy_j}$$

polares Rechteck

$$\Phi\big([r_1,r_2] \times [\varphi_1,\varphi_2]\big)$$





$$\int f\, dA \approx \sum_{ij} \tilde f(r_i, \varphi_j)\, \underbrace{\Delta A_{ij}}_{\approx r\, \Delta r_i\, \Delta\varphi_j}$$

pol. Re. hat Flächeninhalt

$$A = \frac{(\varphi_2 - \varphi_1)}{2\cancel{\pi}} (r_2^2 - r_1^2)\cancel{\pi}$$

$$r_2 = r + \Delta r, \quad r_1 = r, \quad \varphi_2 = \varphi + \Delta\varphi, \quad \varphi_1 = \varphi$$

$$\Rightarrow A = \Big(\underbrace{(r+\Delta r)^2}_{\cancel{r^2} + 2r\Delta r + \Delta r^2} - \cancel{r^2}\Big)\, \Delta\varphi\, \frac{1}{2} = r\, \Delta r\, \Delta\varphi + \frac{1}{2} \Delta r^2 \Delta\varphi$$

$$\approx r\, \Delta r\, \Delta\varphi$$

$$\sum_{ij} \tilde{f}(r_i, \varphi_j)\, r_i \Delta r_i \Delta \varphi_j$$

$$\approx \int_0^\infty \left(\int_{-\pi}^{\pi} r \tilde{f}(r,\varphi)\, d\varphi \right) dr$$

<u>Transformationsformel</u> für Integrale in Polarkoo.:

$$\int d(x,y)\, f(x,y) = \int_0^\infty dr \int_{-\pi}^{\pi} d\varphi\, r \tilde{f}(r,\varphi)$$

<u>Merke</u> "$d(x,y) = r\, dr\, d\varphi$" $\qquad \overbrace{\int_{-\pi}^{\pi} d\varphi}^{2\pi}$

<u>Bsp</u> $\quad f(x,y) = e^{-x^2-y^2}$ $\qquad \int_0^\infty dr \left(\int_{-\pi}^{\pi} r e^{-r^2} d\varphi \right)$

$\qquad \tilde{f}(r,\varphi) = e^{-r^2}$

$$\int_{\mathbb{R}^2} f(x,y)\, d(x,y) = \int_{-\pi}^{\pi} d\varphi \int_0^\infty dr\, r e^{-r^2}$$

$$= 2\pi \int_0^\infty dr\, r e^{-r^2}$$

$$= 2\pi \int_0^\infty dr \left(-\tfrac{1}{2}\right) \frac{d}{dr}\left(e^{-r^2}\right)$$

$$= -\pi \left[e^{-r^2} \right]_0^\infty \quad \text{d.h.} \quad -\pi \lim_{b\to\infty} \underbrace{\left[e^{-r^2} \right]_0^b}_{\underbrace{e^{-b^2}}_{\to 0} - 1}$$

$$= -\pi(-1) = \underline{\underline{\pi}}$$

Andererseits $\quad \int_{\mathbb{R}^2} \underbrace{d(x,y)}_{dx\,dy} e^{-x^2} e^{-y^2}$

$$= \int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} dy\, e^{-x^2} e^{-y^2} \right) = \left(\int_{\mathbb{R}} dx\, e^{-x^2} \right) \left(\int_{\mathbb{R}} dy\, e^{-y^2} \right)$$

$$= \left(\int_{\mathbb{R}} dx\, e^{-x^2} \right)^2 \Rightarrow \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$