Transformationsformel

$=$ Substitutionsregel im $\mathbb{R}^n$

$\Phi: \underset{\mathbb{R}^n}{\cap}U \longrightarrow \underset{\mathbb{R}^n}{\cap}G$ Diffeo

$f: G \to \mathbb{R}$

$$\int_G d^n \underline{y}\; f(\underline{y}) = \int_U d^n \underline{x}\; f(\Phi(\underline{x})) \cdot \text{Faktor}$$

"substituiere $\underline{y} = \Phi(\underline{x})$"

Substitutionsregel im $\mathbb{R}^1$:

$$\int_{\Phi(a)}^{\Phi(b)} dy\; f(y) = \int_a^b dx\; f(\Phi(x)) \cdot \overbrace{\Phi'(x)}^{\text{Faktor}}$$

"$dy = \Phi'(x)\, dx$"

"substituiere $y = \Phi(x)$"

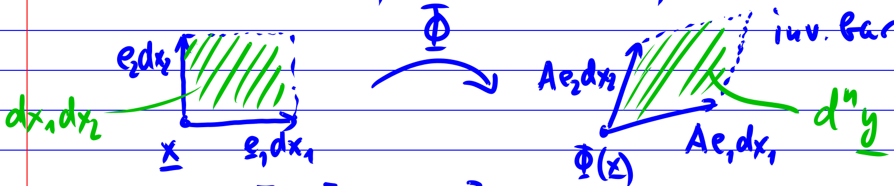
Polarkoo: $d^2\underline{x} = dr\, d\varphi$, Faktor $= r$

$d(x,y) = d(x_1,x_2) = d(r,\varphi)$

allg. $\mathbb{R}^n$: Faktor $=$ "$\dfrac{\mathrm{Vol}\,\Phi([x_1,x_1+dx_1]\times\cdots\times[x_n,x_n+dx_n])}{dx_1\cdots dx_n}$"

<u>Erster Fall:</u> Φ ist lin. Abb,

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(\underline{x}) = A\,\underline{x}, \quad A \in GL(n,\mathbb{R})$$ inv.bar



$[x_1, x_1+dx_1] \times [x_2, x_2+dx_2]$

$$d^n\underline{y} = \mathrm{Vol} = \left|\det\left(Ae_1 dx_1, \ldots, Ae_n dx_n\right)\right|$$

$$= \left|\det(A)\right| \underbrace{dx_1 \cdots dx_n}_{d^n \underline{x}}$$

<u>Prop 10.15</u> Sei $f \in C_0(\mathbb{R}^n)$, $A \in GL(n,\mathbb{R})$. Dann $f \circ A \in C_0(\mathbb{R}^n)$, und

$$\int_{\mathbb{R}^n} d^n\underline{y}\; f(\underline{y}) = \int_{\mathbb{R}^n} d^n\underline{x}\; f(A\underline{x})\,|\det A| .$$

$(\underline{y} = A\underline{x})$

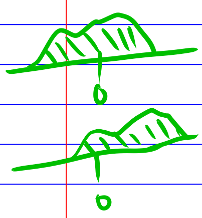
<u>Bew</u> für $n=2$

Sei zunächst $A = R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$ obere Dreiecks-matrix

re. Seite =

$$\int_{\mathbb{R}^2} d^2\underline{x}\; f(R\underline{x})\,|r_{11} r_{22}| = \qquad (r_{11} \neq 0)$$

$$\int_{-\infty}^{\infty} dx_2 \left(\int_{-\infty}^{\infty} dx_1\, |r_{11} r_{22}|\, f(r_{11}x_1 + r_{12}x_2,\, r_{22}x_2) \right)$$



translations-invariant

$$= \int_{-\infty}^{\infty} dx_2\, |r_{11} r_{22}| \int_{-\infty}^{\infty} dx_1\, f(r_{11}x_1, r_{22}x_2)$$

$$
\left.\begin{array}{l}\text{Subst. in } \mathbb{R}^1 \\ y_1 = r_{11} x_1\end{array}\right| = \int_{-\infty}^{\infty} dx_2 \, |r_{11} r_{22}| \int_{-\infty}^{\infty} \frac{dy_1}{\cancel{|r_{11}|}} f(y_1, r_{22} x_2)
$$

$$
\left. y_2 = r_{22} x_2 \right| = \int_{-\infty}^{\infty} \frac{dy_2}{\cancel{|r_{22}|}} \cancel{|r_{22}|} \int_{-\infty}^{\infty} dy_1 f(y_1, y_2)
$$

$$
= \int_{\mathbb{R}^2} d^2\underline{y} \; f(\underline{y}) = \text{li. Seite}
$$

Ebenso für $A = L$ untere Dreiecksmatrix.

Allg. A lässt sich als LR schreiben

(L = untere Dr.m., R = ob. Dr.m.)

Gauß-Alg.: $\quad LR$

$$
L = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}
$$

$$
\text{Also } \underbrace{\int_{\mathbb{R}^2} d^2\underline{x} \; f \circ A \; |\det A|}_{\text{re Seite} =} = \int_{\mathbb{R}^2} d^2\underline{x} \; f \circ L \circ R \; \overbrace{|\det L|}^{|\det R|\,|\det L|}
$$

$$
\overset{\underline{y} = R\underline{x}}{=} \int_{\mathbb{R}^2} d^2\underline{y} \; f \circ L(\underline{y}) \; |\det L|
$$

$$
\overset{\underline{z} = L\underline{y}}{=} \int_{\mathbb{R}^2} d^2\underline{z} \; f(\underline{z}) = \text{li. Seite}
$$

$\underline{z} = A\underline{x}$ $\quad\square$

Bsp. $A \in O(n)$ Rotation / Spiegelung

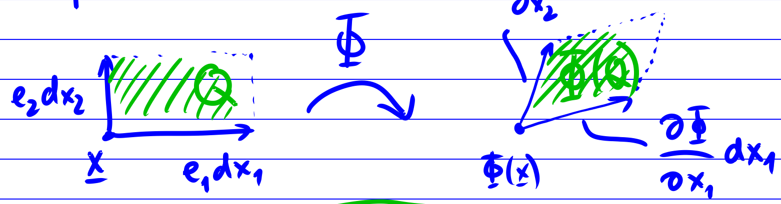
$\Rightarrow \det A = \pm 1$, $|\det A| = 1$

$$
\Rightarrow \int_{\mathbb{R}^n} d^n\underline{y} \; f(\underline{y}) = \int_{\mathbb{R}^n} d^n\underline{x} \; f(A\underline{x})
$$

invariant unter orth. A.

Allg. Fall: Φ Diffeo

infinitesimaler Quader



$$\mathrm{Vol}\ \Phi(Q) = \left| \det \left(D\Phi|_x \right) \right| dx_1 \cdots dx_n$$

Jacobi-Determinante = Funktionaldet.

$D\Phi|_x$: Jacobi-Matrix = Funktionalmatrix

Bsp Polarkoo $\Phi(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

$$D\Phi = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\det D\Phi = r\cos^2\varphi + r\sin^2\varphi = r$$

10.16 Transformationssatz für Integrale

(Subst.regel in $\mathbb{R}^n$)

Seien $U, G \subset \mathbb{R}^n$ offen, $\Phi : U \to G$ Diffeo, $f \in C_0(G)$. Dann $f \circ \Phi \in C_0(U)$ und

$$\int_G d^n y\ f(y) = \int_U d^n x\ f(\Phi(x)) \left| \det D\Phi|_x \right| .$$

Bem 10.17 $\quad n = 1$

$$\int_{\Phi(a)}^{\Phi(b)} dy\, f(y) = \int_a^b dx\, f(\Phi(x)) \quad \Phi'(x)$$

falls ~~ob~~ stets $\Phi'(x) > 0$

oder stets $\Phi'(x) < 0$

dann $\Phi(b) < \Phi(a)$, also

$$\left(\int_{\Phi(a)}^{\Phi(b)} = - \int_{\Phi(b)}^{\Phi(a)} = - \int_{[\Phi(b), \Phi(a)]} \right)$$