Nochmal Matrix-Exponential

Konkrete Bsp.e für e^{At}

Erinnerung: $\dot{\underline{x}} = A\underline{x}$

hat Lsgen $\underline{x}(t) = e^{At}\underline{x}_0$

Es gilt $\underline{e^{SBS^{-1}}} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{(SBS^{-1})^k}_{= SB^kS^{-1}}$

$$= S\left(\sum_{k=0}^{\infty} \frac{1}{k!} B^k\right) S^{-1} = \underline{S e^B S^{-1}}.$$

<u>Bem 9.12</u>

Falls A diag.bar, also $A = SDS^{-1}$ (oder $S^{-1}AS = D$) mit D diagonal, dann

$$e^{At} = e^{S(Dt)S^{-1}} = S e^{Dt} S^{-1}.$$

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix},\ D^2 = \begin{pmatrix} d_1^2 & & \\ & \ddots & \\ & & d_n^2 \end{pmatrix},\ D^k = \begin{pmatrix} d_1^k & & \\ & \ddots & \\ & & d_n^k \end{pmatrix}$$

$$\Rightarrow e^{Dt} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k D^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \begin{pmatrix} d_1^k & & \\ & \ddots & \\ & & d_n^k \end{pmatrix}$$

$$= \sum_{k=0}^{\infty} \begin{pmatrix} \frac{1}{k!} t^k d_1^k & & \\ & \ddots & \\ & & \frac{1}{k!} t^k d_n^k \end{pmatrix}$$

$$= \begin{pmatrix} e^{td_1} & & \\ & \ddots & \\ & & e^{td_n} \end{pmatrix}.$$

Anders gesagt:

Stelle Vektoren in $\mathbb{R}^n$ in Basis {Spalten von S} dar:

$$\underline{x} = S\,\underline{y}$$

$$\Rightarrow S\dot{\underline{y}} = \dot{\underline{x}} = A\underline{x} = AS\underline{y}$$

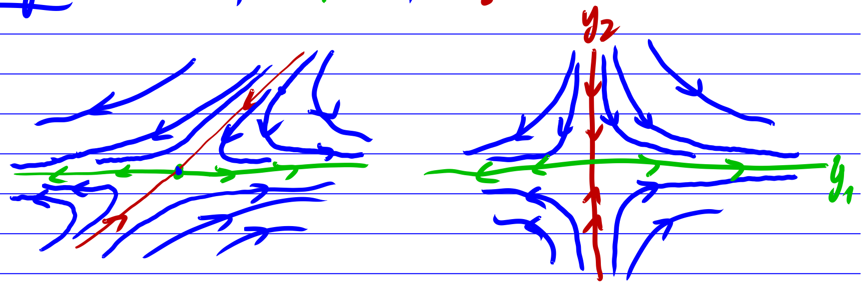
$$\Rightarrow \dot{\underline{y}} = S^{-1}AS\underline{y} = D\underline{y}$$

also $\dot{y}_j = d_j\, y_j$

$$\Rightarrow y_j(t) = e^{d_j t}\, y_{0j}\,, \quad \underline{y}_0 = S^{-1}\underline{x}_0$$

$$\underline{x}(t) = S\,\underline{y}(t).$$

Bsp $n = 2$, $d_1 > 0$, $d_2 < 0$



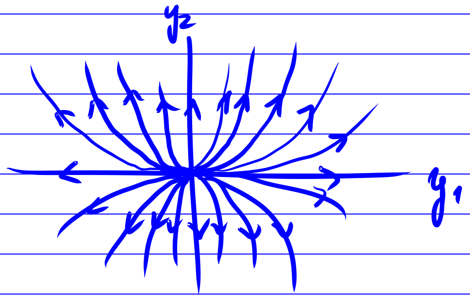
Bahnkurven $y_2(y_1) = ?$

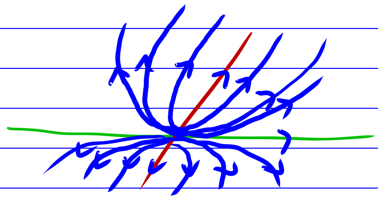
$$y_1(t) = e^{d_1 t}\, y_{01}\,, \quad y_2(t) = e^{d_2 t}\, y_{02}$$

$$y_2(t) = (\text{const.})\,\frac{1}{|y_1(t)|^{|d_2|/d_1}}$$

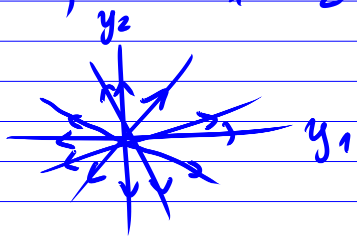
Bsp $n=2$, $0<d_1<d_2$

$$y_2(t) = (\text{const.})\,|y_1(t)|^{d_2/d_1}$$





Bsp $n=2$, $0<d_1=d_2$



Bsp $n=2$, $0=d_1<d_2$

$y_1(t) = \text{const.} = y_{01}$

$y_2(t) = e^{d_2 t}\, y_{02}$