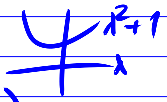
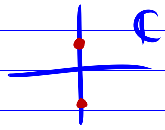
e^{At} falls A komplex diag.bar

z.B. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$P_A(\lambda) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$



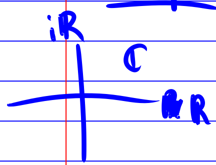
$$= \lambda^2 - i^2 = (\lambda + i)(\lambda - i)$$



Wenn $A \in M(n, \mathbb{R})$, dann hat $P_A(\lambda)$ reelle Koeffizienten, und NSTen entweder reell oder in konjugierten Paaren, $\lambda_0 \in \mathbb{C}$ und $\bar{\lambda}_0 \in \mathbb{C}$.

Bew $P_A(\bar{\lambda}) = \overline{P_A(\lambda)}$ $\square$

Bsp in $\mathbb{C}^n$ Eigenraum V, $\dim_{\mathbb{C}} V = 1$

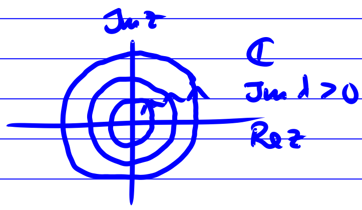


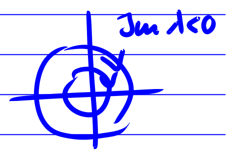
$A|_V = \lambda$. Wähle Basis $\underline{v} \in V$,

$V \cong \mathbb{C}$, $\underline{z}(t)$, $\dot{\underline{z}} = A\underline{z}$ in $\mathbb{C}^n$

$$\underline{z}(t) = e^{\lambda t}\underline{z}_0 = e^{\mathrm{Re}(\lambda)t}\, e^{i\,\mathrm{Im}(\lambda)t}\, \underline{z}_0$$

Bsp $\mathrm{Re}\,\lambda = 0$





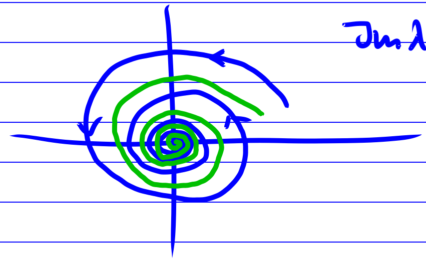
Bsp $\operatorname{Re}\lambda < 0$

exp. Spirale

$r = (\text{const.})\, e^{a\varphi}$

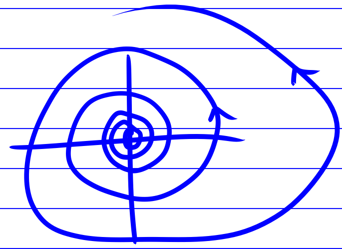
$a = \operatorname{Re}\lambda$

$z(t) \xrightarrow{t\to\infty} 0$



$\operatorname{Im}\lambda > 0$

Bsp $\operatorname{Re}\lambda > 0$



Reelle Lsgen

Eigenraum V_1 mit EW λ_1 $\quad \dim_{\mathbb{C}} V_1 = 1$

V_2 mit EW $\lambda_2 = \overline{\lambda_1}$, $\dim_{\mathbb{C}} V_2 = 1$

Wähle Basis $\underline{v}_1 \in V_1$, $\underline{v}_2 \in V_2$ $\quad \lambda_1 \notin \mathbb{R}$

$\{\underline{v}_1, \underline{v}_2\}$ Basis von $V_1 \oplus V_2$

$$\underline{z}(t) = z_1(t)\,\underline{v}_1 + z_2(t)\,\underline{v}_2$$

$$z_i(t) = e^{\lambda_i t} z_{0i}$$

$$\Rightarrow \underline{z}(t) = \sum_{i=1}^{2} e^{\lambda_i t} z_{0i}\,\underline{v}_i \in \mathbb{C}^n$$

Beh $\underline{z}(t) \in \mathbb{R}^n \;\forall t \iff z_{02}\underline{v}_2 = \overline{z_{01}\underline{v}_1}$

Bew "$\Leftarrow$": Wg. $\lambda_2 = \overline{\lambda_1}$ und

$z + \bar{z} \in \mathbb{R} \quad \forall z \in \mathbb{C}$

"$\Rightarrow$": $t = 0 \Rightarrow \operatorname{Im}(z_{01}\underline{v}_1) = -\operatorname{Im}(z_{02}\underline{v}_2)$

$t = \frac{\pi}{2\operatorname{Im}\lambda_1} \overset{\lambda_2 = \overline{\lambda_1}}{\Longrightarrow} \operatorname{Im}(i z_{01}\underline{v}_1) = -\operatorname{Im}(-i z_{02}\underline{v}_2)$

$\iff \operatorname{Re}(z_{01}\underline{v}_1) = \operatorname{Re}(z_{02}\underline{v}_2)$ $\square$

Dann $\underline{z}(t) = 2\operatorname{Re}\left(e^{\lambda_1 t} z_{01}\underline{v}_1\right)$

$$= 2\, e^{\mathrm{Re}(\lambda_1)t}\, \mathrm{Re}\left(\underbrace{e^{i\,\mathrm{Im}(\lambda_1)t}}_{\cos + i \sin}\, z_{01}\, \underline{v}_1 \right)$$

$$= 2\, \overbrace{e^{\mathrm{Re}(\lambda_1)t} \cos\left(\mathrm{Im}(\lambda_1)t\right)}^{y_1(t)}\, \mathrm{Re}\left(z_{01}\, \underline{v}_1\right)$$

$$+ 2\, \underbrace{e^{\mathrm{Re}(\lambda_1)t} \sin\left(\mathrm{Im}(\lambda_1)t\right)}_{y_2(t)}\, \mathrm{Re}\left(i z_{01}\, \underline{v}_1\right)$$