e^{At} falls A nicht komplex diag.bar

$\Rightarrow$ Jordan-Normalform in $\mathbb{C}^n$

d.h. $A = S J S^{-1}$

$$J = \left[\begin{array}{ccccc|cccc|c} \lambda_1 & 1 & & & & & & & & \\ & \lambda_1 & 1 & & & & & & & \\ & & \lambda_1 & 1 & & & & & & \\ & & & \ddots & 1 & & & & & \\ & & & & \lambda_1 & & & & & \\ \hline & & & & & \lambda_2 & 1 & & & \\ & & & & & & \lambda_2 & \ddots & & \\ & & & & & & & \ddots & 1 & \\ & & & & & & & & \lambda_2 & \\ \hline & & & & & & & & & \ddots \end{array}\right]$$

$$= \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \ddots \end{bmatrix}, \quad e^{Jt} = \begin{bmatrix} \boxed{e^{J_1 t}} & & \\ & \boxed{e^{J_2 t}} & \\ & & \ddots \end{bmatrix}$$

Bsp $J = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$, $\dot{\underline{y}} = J \underline{y}$

$$\Leftrightarrow \quad \frac{dy_1}{dt} = 2y_1 + y_2$$

$$\frac{dy_2}{dt} = 2y_2$$

$$\Rightarrow \quad y_2(t) = e^{2t}\, y_{02}$$

$$\frac{dy_1}{dt} = 2y_1 + e^{2t} y_{02}$$

hom.: $\frac{du}{dt} = 2u, \Rightarrow u(t) = e^{2t} u_0$

Var. der Konst.: $y_1(t) = e^{2t} u_0(t)$ Ansatz

$$\Rightarrow \quad \dot{y}_1(t) = 2y_1(t) + e^{2t}\, \dot{u}_0(t)$$

einsetzen $\cancel{2y_1(t)} + e^{2t} y_{02} = \cancel{2y_1(t)} + e^{2t} \dot{u}_0(t)$

$$\Rightarrow \quad \dot{u}_0(t) = y_{02}$$

$$\Rightarrow \quad u_0(t) = y_{02}\, t + u_{00}$$

$$\Rightarrow y_1(t) = t e^{2t} y_{02} + e^{2t} u_{00}$$

AW $y_1(0) = u_{00} = y_{01}$

Also $\underline{y}(t) = \begin{pmatrix} e^{2t} y_{01} + t e^{2t} y_{02} \\ e^{2t} y_{02} \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{pmatrix}}_{e^{Jt}} \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$$

Allg. $J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$, $\dot{\underline{y}} = J\underline{y}$

$$\Leftrightarrow \quad \frac{dy_1}{dt} = \lambda y_1 + y_2$$

$$\frac{dy_2}{dt} = \lambda y_2 + y_3$$

$$\vdots$$

$$\frac{dy_n}{dt} = \lambda y_n$$

$$\Rightarrow \quad y_n(t) = e^{\lambda t} y_{0n} \quad \text{etc.}$$

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exp-Reihe $e^{Jt} = ?$

$J = \lambda E + N$, $\quad N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & 0 & \ddots & \ddots & \\ & & & \ddots & 1 & 0 \\ & & & & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix},$$

$$N^k = \begin{bmatrix} \overbrace{0 \cdots 0}^{k} & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} k \end{matrix}$$

$$e^{Jt} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \underbrace{J^k}_{(\lambda E + N)^k}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \underbrace{(\lambda E)^{k-j}}_{\lambda^{k-j} E} N^j$$

$$\Rightarrow e^{Jt} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{1}{j!\,(k-j)!} t^k \lambda^{k-j} N^j$$

$$= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \text{———} \;{}''\; \text{———}$$

$$\overset{m=k-j}{=} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{j!\,m!} t^{m+j} \lambda^m N^j$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} t^j \underbrace{N^j}_{\substack{=0 \\ \forall j \geq n}} \underbrace{\sum_{m=0}^{\infty} \frac{1}{m!} t^m \lambda^m}_{e^{\lambda t}}$$

$$= e^{\lambda t} \sum_{j=0}^{n-1} \frac{t^j}{j!} N^j$$

$$= e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 & \cdots & \frac{1}{(n-1)!}t^{n-1} \\ & 1 & t & \frac{1}{2}t^2 & \ddots & \vdots \\ & & 1 & t & \ddots & \frac{1}{3!}t^3 \\ & & & 1 & \ddots & \frac{1}{2}t^2 \\ & & & & \ddots & t \\ & & & & & 1 \end{bmatrix}$$

Bsp $J = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$, $e^{Jt} = e^{2t} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$

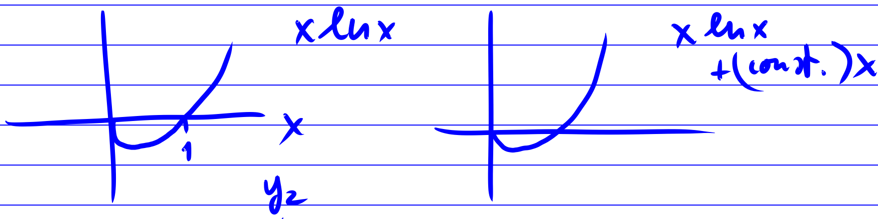
$\dot{\underline{y}} = J\underline{y}$

$\underline{y}(t) = e^{Jt}\underline{y}_0 = e^{2t} \begin{bmatrix} y_{01} + t\,y_{02} \\ y_{02} \end{bmatrix}$

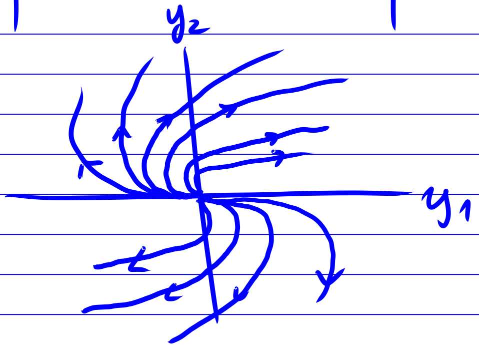
Bahnen $y_1(y_2) = ?$

$$e^{2t} = \frac{y_2(t)}{y_{02}} \Rightarrow y_1(t) = \frac{y_{01}}{y_{02}} y_2(t) + \frac{1}{2} \ln\left(\frac{y_2(t)}{y_{02}}\right) y_2(t)$$

also $y_1 = \frac{1}{\lambda} y_2 \ln y_2 + (\text{const.})\, y_2$



also



Bsp $\dot{\underline{y}} = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \underline{y} \Leftrightarrow \begin{cases} \dot{y}_1 = 2y_1 + y_2 \\ \dot{y}_2 = 2y_2 \end{cases}$

$$\Rightarrow \ddot{y}_1 = 2\dot{y}_1 + \underbrace{\dot{y}_2}_{2y_2} = 4\dot{y}_1 - 2(2y_1 + y_2) + 2y_2$$

$$= 4\dot{y}_1 - 4y_1$$

exp-Ansatz $\Rightarrow \quad \lambda^2 = 4\lambda - 4$

$\Leftrightarrow \quad (\lambda - 2)^2 = 0$

$\Rightarrow$ Fund. Lsg. en $\quad y_1(t) = e^{2t}$

und $\quad y_1(t) = t e^{2t}$

$\Rightarrow$ allg. Lsg. $y_1(t) = \underline{a e^{2t} + b t e^{2t}}$