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Geometry in Physics

Homework Assignment # 1

Problem 1: The sphere as a differentiable manifold

Find a C^1-atlas for the unit sphere in $\mathbb{R}^3$, i.e. for

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}\,.$$

Show that the charts of your atlas are compatible.
Hint: You need at least two charts. You can use, for example, spherical coordinates, the stereographic projection or a generalization of the atlas of S^1 given in the lecture.

Problem 2: Equivalent atlases and differentiable structures

Prove that two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on a topological manifold M are equivalent if and only if the identity map

$$\mathrm{Id} : (M, [\mathcal{A}_1]) \to (M, [\mathcal{A}_2]), \quad x \mapsto x\,,$$

is a diffeomorphism with respect to the differentiable structures $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$.

Hint: There is not much to do here beyond understanding and somewhat rephrasing the definitions of the involved notions.

Problem 3: Inequivalent atlases

Find two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ for $\mathbb{R}$ that are not equivalent! With the atlases you found, are $M = (\mathbb{R}, [\mathcal{A}_1])$ and $N = (\mathbb{R}, [\mathcal{A}_2])$ diffeomorphic, i.e. does there exist a diffeomorphism $\phi : M \to N$?

Hint: Start by looking for a differentiable homeomorphism $f : \mathbb{R} \to \mathbb{R}$ that is **not** *a diffeomorphism.*

Problem 4: Induced differentiable structure

Let M be a differentiable manifold, N a topological space, and $f : M \to N$ a homeomorphism. Show that N is a topological manifold and that there exists a unique differentiable structure on N such that f is a diffeomorphism.

Please upload your written solutions in Ilias until Monday, November 16, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.