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Geometry in Physics

Homework Assignment # 10

Problem 37: Energy momentum tensor of a scalar field

Let (M, g) be a Lorentzian manifold. A scalar field on M is a smooth function $f : M \to \mathbb{R}$. Its energy momentum tensor $T_f \in \mathcal{T}^0_2(M)$ is

$$T_f(X, Y) = df(X) \cdot df(Y) - \frac{1}{2} g(\mathrm{grad} f, \mathrm{grad} f) g(X, Y).$$

The (generalized) divergence is the map $\mathrm{div} := C_{12} \circ \nabla : \mathcal{T}^0_2(M) \to \mathcal{T}^0_1(M)$.
Show that $\mathrm{div} T_f = 0$ if and only if the scalar field f it is a solution of the free wave equation $\Box f = 0$.

Problem 38: The Hessian of a function

Let $M = \mathbb{R}^n$ be equipped with the pseudo-Riemannian metric g_ν and $f : M \to \mathbb{R}$ be the function from Example 9.22. Compute $\nabla^2 f$ (in terms of g_ν) and $\mathrm{tr}_{g_\nu} \nabla^2 f$.

Problem 39: Totally umbilic hypersurfaces

Let M and f be as in Exercise 37. Show that for $c \neq 0$, the pseudo-Riemannian hypersurfaces $f^{-1}(c) \subset M$ are totally umbilic.

Problem 40: Covariant derivative and reparametrization

Let (M, g) be a pseudo-Riemannian manifold, $I, J \subset \mathbb{R}$ intervals, $c : I \to M$ a smooth curve, $\varphi : J \to I$ a diffeomorphism and $\tilde{c} = c \circ \varphi$ be a reparametrization of c. Let $\frac{\nabla}{ds}$ be the covariant derivative along c and $\frac{\nabla}{dt}$ the covariant derivative along $\tilde{c}$. Show that

$$\frac{\nabla}{dt}(X \circ \varphi) = \varphi' \cdot (\frac{\nabla}{ds} X) \circ \varphi.$$

Please upload your written solutions in Ilias until Monday, February 08, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.