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Winter term 2020/21
February 8, 2020

Geometry in Physics

Homework Assignment # 11

Problem 41: Reparametrization of geodesics

Let (M, g) be a semi-Riemannian manifold. A curve $c : I \to M$ is called pregeodesic if there exists a function $f : I \to \mathbb{R}$ with $f(t) \neq 0$ for all $t \in I$ such that

$$\frac{\nabla}{dt} c' = f \cdot c'.$$

Show that c is a pregeodesic if and only if there exists a reparametrization $\varphi : J \to I$ such that $c \circ \varphi$ is a geodesic.
Hint: Use Problem 40.

Problem 42: Geodesics of the Poincaré half-plane

Let $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric $g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$.
Show that the image of each maximal geodesic in (M, g) is either a vertical line (with x constant) or a half-circle centered at the x-axix.
Hint: Use Problem 41

Problem 43: A local expression for the Riemann tensor

Let (M, g) be a pseudo-Riemannian manifold with curvature tensor R. Consider the local coefficients of R with respect to a chart (V, φ), given by

$$R(\partial_{q_i}, \partial_{q_j})\partial_{q_k} = R^l_{ijk}\partial_{q_l}.$$

Give a formula for R^l_{ijk} in terms of the Christoffel symbols. Show in addition that

$$R^l_{ijk} = \frac{1}{2} g^{lm} \left(\frac{\partial^2 g_{jm}}{\partial q_i \partial q_k} - \frac{\partial^2 g_{jk}}{\partial q_i \partial q_m} - \frac{\partial^2 g_{im}}{\partial q_j \partial q_k} + \frac{\partial^2 g_{ik}}{\partial q_j \partial q_m} \right) + Q(g^{-1}, g^{-1}, \partial g, \partial g),$$

where Q denotes a schematic expression which is bilinear in the inverse matrix g^{ij} and in first partial derivatives of the metric $\partial_{q_k} g_{ij}$.

Problem 44: Curvature and isometries

Prove Lemma 13.3 which roughtly asserts that the curvature tensor is preservedünder isometries.

Please upload your written solutions in Ilias until Monday, February 15, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.