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Winter term 2020/21
February 15, 2020

Geometry in Physics

Homework Assignment # 12

Problem 45: Curvature of the Poincaré half-plane

Prove that the Poincaré half-plane from Problem 42 has constant sectional curvature -1.

Problem 46: Second fundamental form and the Hessian

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $f(0) = 0$ and $Df|_0 = 0$. Consider the Riemannian submanifold

$$M := \mathrm{graph} f = \left\{ (x, f(x)) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n \right\}$$

of $\mathbb{R}^{n+1}$ equipped with the Euclidean metric.

- Find a unit normal field of M.
- Show that the second fundamental form of M at the origin $0 \in M \subset \mathbb{R}^{n+1}$ is given by

$$\Pi(v, w) = \mathrm{Hess}(f)(v, w) \cdot (0, \dots, 0, 1)$$

Hint: Note that $T_0 M = \{(x, 0) \mid x \in \mathbb{R}^n\} = \mathbb{R}^n$ *and* $(T_0 M)^\perp = \{(0, y) \mid y \in \mathbb{R}\}$.

Problem 47: Curvature of quadratic forms

The function $f : \mathbb{R}^n \to \mathbb{R}$, given by $f : x \mapsto \frac{1}{2} \sum_{i=1}^n a_i (x^i)^2$, (with some $a_i \in \mathbb{R}$) fulfills the condition of Problem 43.

- Compute for each $1 \leq i < j \leq n$ the sectional curvature K_{ij} of the $x^i - x^j$ plane in $T_0 M$.
- Sketch a picture of $M \cap \mathrm{span}(x^i, x^j, x^{n+1})$ in $\mathbb{R}^3 := \mathrm{span}(x^i, x^j, x^{n+1})$ for different values of a_i and a_j.

Problem 48: Pointwise constant sectional curvature

Let (M, g) be a pseudo-Riemannian manifold and suppose there exists a function $K : M \to \mathbb{R}$ such that for all $x \in M$, the sectional curvature of each 2-plane $\sigma \subset T_x M$ is equal to $K(x)$. Prove that if $\dim(M) > 2$, K must be a constant function.
Hint: Use Remark 13.15 and the second Bianchi identity.

Please upload your written solutions in Ilias until Monday, February 22, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.