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Winter term 2020/21
November 16, 2020

Geometry in Physics

Homework Assignment # 2

Problem 5: Diffeomorphisms

Show that the manifolds S^1 and $\mathbb{R}/\mathbb{Z}$ constructed in the lecture are diffeomorphic, i.e. find a bijection between these two manifolds and show that it is a diffeomorphism.

Problem 6: Tangent vectors as derivations

In the lecture it was shown that every tangent vector $v \in T_xM$ to a manifold M at a point $x \in M$ defines a derivation D_v at that point. In this problem we motivate that these are actually all derivations at a point $x \in M$, i.e. that every derivation D at x is of the form $D = D_v$ for some tangent vector $v \in T_xM$, by proving the corresponding statement for $M = \mathbb{R}^n$.

Recall that for a n-dimensional smooth manifold M a **derivation at** $x \in M$ is a linear map $D : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ that satisfies the product rule

$$D(fg) = D(f)\, g(x) + f(x)\, D(g) \qquad \text{for all } f, g \in C^\infty(M, \mathbb{R})\,. \tag{1}$$

Now let $M = \mathbb{R}^n$. Prove that every derivation D at $x \in \mathbb{R}^n$ is of the form D_v for some tangent vector $v \in T_xM \cong \mathbb{R}^n$, where

$$D_v(f) = v \cdot \nabla f(x)\,. \tag{2}$$

Hint: First show that (1) *implies that* $D(1) = 0$*, where* $1 \in C^1(M)$ *denotes the function that is constant equal to one. Conclude that* $D(f) = 0$ *for any constant function. Now look at the first order Taylor expansion of* f *at the point* x *with explicit remainder term in integral form to conclude that* $D(f)$ *is of the form* (2)*.*

Problem 7: Polar coordinates

We equip the manifold $M = \mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 \,|\, x_2 = 0 \text{ and } x_1 \leq 0\} \subset \mathbb{R}^2$ with the charts (M, φ_1) and (M, φ_2) where $\varphi_1(x) = (x_2, x_1)$ and

$$\varphi_2^{-1} : (0, \infty) \times (-\pi, \pi) \to M\,, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta)\,.$$

Show that φ_1 and φ_2 are compatible and express the coordinate vectors ∂_{q_1} and ∂_{q_2} associated with φ_1 as well as the coordinate vectors ∂_r and ∂_θ associated with φ_2 at a point $x \in M$ in terms of the canonical basis vectors e_1 and e_2 in $T_xM \cong \mathbb{R}^2$.

Problem 8: Inverse function theorem on manifolds

Let M and N be differentiable manifolds of dimension n and $f \in C^1(M, N)$. Assume that at some $x \in M$ the differential $Df|_x : T_xM \to T_{f(x)}N$ is an isomorphism. Use the inverse function theorem on $\mathbb{R}^n$ in order to show that there exists an open neighbourhood $U \subset M$ of x, such that $V := f(U)$ is open and $f : U \to V$ is a diffeomorphism.

Please upload your written solutions in Ilias until Monday, November 23, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.