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Geometry in Physics

Homework Assignment # 3

Problem 9: The orthogonal matrices as a manifold Show that the orthogonal matrices $O(n) := \{Q \in \mathrm{GL}(n) \,|\, Q^T Q = \mathrm{Id}\}$ form a $\frac{n(n-1)}{2}$-dimensional submanifold of the manifold of $n \times n$-matrices $\mathrm{mat}(n) \cong \mathbb{R}^{n^2}$. Show also that

$$T_Q O(n) = \{B \in \mathrm{mat}(n) \,|\, (Q^{-1}B)^T = -Q^{-1}B\}\,,$$

and hence, in particular,

$$T_{\mathrm{Id}} O(n) \;=\; \{B \,|\, B^T = -B\} \;=:\; \mathrm{skew}(n).$$

Hint: Find a suitable map $F : \mathrm{mat}(n) \to \mathrm{sym}(n)$ into the symmetric matrices sym(n) with $F^{-1}(\{0\}) = O(n)$ and apply Proposition 2.21 of the Lecture.

Problem 10: The Lie derivative of functions

Let M be a differentiable manifold. Show the following properties of the Lie-derivative:

(i) $L_X(f+g) = L_X f \,+\, L_X g$

(ii) $L_X(fg) = (L_X f)g \,+\, f(L_X g)$

(iii) $L_{\alpha X + \beta Y}(f) = \alpha\, L_X f \,+\, \beta\, L_Y f$

for all $f, g, \alpha, \beta \in C^\infty(M)$ and $X, Y \in \mathcal{T}_0^1(M)$.

Problem 11: The commutator of vector fields

Let M be a differentiable manifold and $X, Y \in \mathcal{T}_0^1(M)$. Show that there exists a vector field $Z \in \mathcal{T}_0^1(M)$ such that

$$L_X \circ L_Y \,-\, L_Y \circ L_X \;=\; L_Z.$$

Hint: You may either use the statement of remark 2.32 in the lecture notes or do an explicit computation using charts. In the latter approach you need to show, however, that the vector field Z does not depend on the choice of charts.

Problem 12: Line integrals of 1-forms

Let M be a smooth manifold, $I = [a, b] \subset \mathbb{R}$ an interval, $\gamma \in C^\infty(I, M)$ a smooth curve, and $\omega \in \mathcal{T}_1^0(M)$ a 1-form. Recall that the integral of ω along γ is the number

$$\int_\gamma \omega := \int_I \gamma^* \omega := \int_a^b (\gamma^* \omega \,|\, e)(t) \, \mathrm{d}t \,,$$

where $\gamma^*\omega$ is the pull-back of ω to I under γ and $e : I \to I \times \mathbb{R}$, $t \mapsto (t, 1)$, is the unit vector field on I. The dual pairing $(\gamma^*\omega \,|\, e) \in C^\infty(I)$ between $\gamma^*\omega \in \mathcal{T}_1^0(I)$ and $e \in \mathcal{T}_0^1(I)$ is to be taken pointwise and defines a smooth function on I that is integrated in the standard Riemannian sense.

(a) For $t \in I$ let $\gamma'(t) := D\gamma|_t \, e(t) \in T_{\gamma(t)}M$. Show that

$$\int_\gamma \omega = \int_a^b \big(\omega(\gamma(t)) \,|\, \gamma'(t)\big) \, \mathrm{d}t \,. \qquad (*)$$

(b) Let $\tilde{I} = [\tilde{a}, \tilde{b}]$ be another interval, $\Phi : \tilde{I} \to I$ a diffeomorphism with $\Phi'(t) > 0$, and $\tilde{\gamma} : \tilde{I} \to M$, $\tilde{\gamma}(t) := (\gamma \circ \Phi)(t)$, a reparametrisation of the curve γ. Show that

$$\int_{\tilde{\gamma}} \omega = \int_\gamma \omega \,.$$

Hint: First show that $\tilde{\gamma}'(t) = \Phi'(t)\, \gamma'(\Phi(t))$ *and then use* $(*)$ *and the usual substitution rule for one-dimensional integrals.*

(c) Now let $f \in C^\infty(M)$ and $\mathrm{d}f \in \mathcal{T}_1^0(M)$ its differential. Prove the fundamental theorem of calculus:

$$\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a)) \,.$$

Hint: First show that $\big(\mathrm{d}f(\gamma(t)) \,|\, \gamma'(t)\big) = \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma(s))\big|_{s=t}$ *and then use the fundamental theorem of calculus for functions on* $\mathbb{R}$.

Please upload your written solutions in Ilias until Monday, November 30, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.