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Geometry in Physics

Homework Assignment # 4

Problem 13: Transformation of coordinates

Let M be a manifold and (V, φ) and $(\tilde{V}, \tilde{\varphi})$ be coordinate neighborhoods with associated coordinate 1-forms $\{dq^i\}_{1\leq i\leq n}$ and $\{d\tilde{q}^j\}_{1\leq j\leq n}$. Show that

$$dq^i|_x = A^i_j d\tilde{q}^j|_x, \qquad \text{with } A^i_j = \partial_j(e^i \circ \varphi \circ \tilde{\varphi}^{-1})|_{\tilde{\varphi}(x)},$$

for all $x \in V \cap \tilde{V}$. Here, e^i denotes the projection onto the ith coordinate. The shorthand notation of this formula is

$$dq^i = \frac{\partial q^i}{\partial \tilde{q}^j} d\tilde{q}^j.$$

Determine the transformation rule for coordinate (r, s)-tensor fields

$$\partial_{q_{i_1}} \otimes \ldots \otimes \partial_{q_{i_r}} \otimes dq^{j_1} \otimes \ldots \otimes dq^{j_s}.$$

Problem 14: The wedge product

Let $\omega \in \Lambda_k$ and $\nu \in \Lambda_p$. Show that

$$\omega \wedge \nu = (-1)^{kp}\, \nu \wedge \omega\,.$$

Hint: You can use basis representations and the property (5.1) of the wedge product of covectors.

Problem 15: The canonical volume form

Let $g \in V^0_2$ be non-degenerate, $(e^j)_{j=1,\ldots,n}$ and $(\tilde{e}^j)_{j=1,\ldots,n}$ bases of V^* with the same orientation, and

$$g = \sum_{i,j=1}^{n} g_{ij}\, e^i \otimes e^j = \sum_{i,j=1}^{n} \tilde{g}_{ij}\, \tilde{e}^i \otimes \tilde{e}^j\,.$$

Show that

$$\sqrt{|\det g_{ij}|}\, e^1 \wedge e^2 \wedge \ldots \wedge e^n = \sqrt{|\det \tilde{g}_{ij}|}\, \tilde{e}^1 \wedge \tilde{e}^2 \wedge \ldots \wedge \tilde{e}^n\,, \qquad (*)$$

i.e. that the canonical volume form ε defined by the expression $(*)$ does not depend on the choice of basis.

Problem 16: The Hodge isomorphism

In this assignment we show that for a non-degenerate and symmetric $g \in V_2^0$ the Hodge operator satisfies

$$* \circ *|_{\Lambda_k} \;=\; (-1)^{k\,(n-k)}\,\mathrm{sgn}(\det g_{ij})\,. \tag{1}$$

(a) First argue that there exists a basis (e^j) of V^* such that the component matrix g_{ij} of $g = g_{ij} e^i \otimes e^j$ is diagonal, i.e. $g_{ij} = 0$ if $i \neq j$.

(b) Let $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ be an ordered k-tuple with $j_i \in \{1, \ldots, n\}$. Now compute $*(e^{j_1} \wedge \cdots \wedge e^{j_k}) := i_{e^{j_1} \wedge \cdots \wedge e^{j_k}} \varepsilon$, where (e^j) is the basis from (a).

(c) Finally show equation (1).

Please upload your written solutions in Ilias until Monday, December 7, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.