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Geometry in Physics

Homework Assignment # 5

Problem 17: Hodge duality in Minkowski space

Let $*$ be the Hodge operator with respect to the Minkowski metric η on $\mathbb{R}^4$. Denoting the canonical coordinates of $\mathbb{R}^4$ with (t, x^1, x^2, x^3) we have $\eta = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i$. Compute the images of the canonical basis vectors of $\Lambda_k(\mathbb{R}^4)$ for each $0 \leq k \leq 4$ under the map $*$.

Hint: Save time by using formula (1) from problem 16!

Problem 18: Restriction of forms to submanifolds

Consider $\mathbb{R}^n$ with the natural coordinates $(x^1, \dots x^n)$. For $k \leq n$, we have the natural embedding $\psi : \mathbb{R}^k \ni (x^1, \dots x^k) \mapsto (x^1, \dots x^k, 0, \dots 0) \in \mathbb{R}^n$. For a given differential form

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda_p(\mathbb{R}^n),$$

compute $\psi^* \omega$.

Problem 19: Naturality of the exterior derivative

Let $f : M \to N$ be a smooth map between smooth manifolds M and N and let $\omega \in \Lambda_p(N)$. Show that

$$f^* \mathrm{d}\omega = \mathrm{d}(f^* \omega)\,.$$

Hint: Look at the computation in the proof of proposition 5.16 in the lecture notes.

Problem 20: Maxwell's equations

Let $*$ be the Hodge operator with respect to the Minkowski metric η on $\mathbb{R}^4$. We assume that the electric field E, the magnetic field B, and the current density J are smooth time-dependent vector fields on $\mathbb{R}^3$, e.g.

$$E : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3\,, \quad (t, x) \mapsto E(t, x)\,.$$

The charge density is a smooth real-valued function $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$.
One now defines corresponding differential forms on Minkowski space $\widetilde{\mathcal{J}}, \mathcal{E} \in \Lambda_1(\mathbb{R}^4)$ and $\mathcal{B}, \mathcal{F} \in \Lambda_2(\mathbb{R}^4)$ by

$$\begin{aligned}
\widetilde{\mathcal{J}} &:= \iota_\eta(\rho\,\partial_t + J \cdot \partial_x) \;=\; -\rho\,\mathrm{d}t \;+\; J_i\,\mathrm{d}x^i, \\
\mathcal{E} &:= E_i\,\mathrm{d}x^i, \\
\mathcal{B} &:= *\,(-B_i\,\mathrm{d}t \wedge \mathrm{d}x^i), \\
\mathcal{F} &:= \mathcal{B} - \mathrm{d}t \wedge \mathcal{E}\,.
\end{aligned}$$

Finally we define the current 3-form as $\mathcal{J} := *\widetilde{\mathcal{J}}$.
Prove the following two equivalences:

$$\frac{\partial B}{\partial t} + \mathrm{curl} E \;=\; 0 \quad \& \quad \mathrm{div} B \;=\; 0 \qquad \Longleftrightarrow \qquad \mathrm{d}\,\mathcal{F} \;=\; 0,$$

$$-\frac{\partial E}{\partial t} + \mathrm{curl} B \;=\; J \quad \& \quad \mathrm{div} E \;=\; \rho \qquad \Longleftrightarrow \qquad \mathrm{d}(*\mathcal{F}) \;=\; \mathcal{J}.$$

Thus, Maxwell's equations have a very simple form when written in terms of differential forms.

Please upload your written solutions in Ilias until Monday, December 14, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.