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Geometry in Physics

Homework Assignment # 56

Problem 21: The continuity equation on Minkowski space

Let $*$ be the Hodge operator with respect to the Minkowski metric η on $\mathbb{R}^4$. Given a time-dependent smooth vector field $J : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and a time-dependent smooth density $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, we define the current 3-form $\mathcal{J} \in \Lambda_3(\mathbb{R}^4)$ as in problem 20. Show that continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div} J = 0$$

is equivalent to

$$\mathrm{d}\,\mathcal{J} \;=\; 0\,.$$

Show, in addition, that the inhomogeneous Maxwell equation

$$\mathrm{d}*\mathcal{F} \;=\; \mathcal{J}$$

has a solution $\mathcal{F}$ if and only if the current $\mathcal{J}$ solves the continuity equation. Is the solution unique?

Problem 22: Characterization of orientability 1

Let M be an oriented manifold of dimension n. Show that there exists a form $\omega \in \Lambda_n(M)$ such that $\omega(x) \neq 0$ for all $x \in M$.
Hint: Define n-forms on coordinate neighbourhoods of an orientable atlas and use a partition of unity to patch them together.

Problem 23: Characterization of orientability 2

Let M be manifold of dimension n and suppose, there exists a form $\omega \in \Lambda_n(M)$ such that $\omega(x) \neq 0$ for all $x \in M$. Show that M is orientable.
Hint: How can we turn an arbitrary atlas into an orientable one?

Problem 24: Integral curves

Consider the vector field $X \in \mathcal{T}_0^1(\mathbb{R}^2)$, given by $X : q = (q_1, q_2) \mapsto (-2q_2, 3q_1)$. Compute its integral curves. Find submanifolds of $\mathbb{R}^2$ on which this vector field is complete.

Please upload your written solutions in Ilias until Monday, January 11, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.