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Geometry in Physics

Homework Assignment # 7

Problem 25: Linearisation of vector fields at fixed-points

Let $X \in \mathcal{T}_0^1(M)$ be a smooth vector field and $x_0 \in M$ a zero of X, i.e. $X(x_0) = (x_0, 0)$. In a chart φ with $\varphi(x_0) = 0$ let

$$X_\varphi(q) = DX_\varphi(0)\, q + \mathcal{O}(\|q\|^2)$$

be the Taylor approximation of $X_\varphi := I \circ \varphi_* X$ at 0. Here Df denotes the usual differential of a function $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e. the Jacobi matrix $Df(q)_{ij} = \partial_{q_j} f_i(q)$. Let ψ be a further chart with $\psi(x_0) = 0$ and $\Phi = \psi \circ \varphi^{-1}$ the transition map.

Show that

$$DX_\psi(0) = D\Phi(0)\, DX_\varphi(0)\, D\Phi(0)^{-1}$$

and conclude that the eigenvalues of the "linearisation" $DX_\varphi(0)$ and their multiplicities do not depend on the chosen chart.

Problem 26: The flow of a vector field

For $a, b \in \mathbb{R}$, consider the matrix $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and define the vector field $X \in \mathcal{T}_0^1(\mathbb{R}^2)$ as $X : q \mapsto A \cdot q$. Compute the flow Φ_t^X of X. For a point $q \in \mathbb{R}^2$, determine the behaviour of $\Phi_t^X(q)$ as $t \to \pm\infty$.

Problem 27: Pullback along a flow

Compute the pullback of the standard metric $dq^1 \otimes dq^1 + dq^2 \otimes dq^2$ on $\mathbb{R}^2$ along the flow Φ_t^X from the previous exercise.

Problem 28: Local formula for the Lie derivative

Let $t \in \mathcal{T}_s^r(M)$ and $X \in \mathcal{T}_0^1(M)$. Let (V, φ) be a local chart with respect to which we expand t and X as

$$t|_V = t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \partial_{q_{i_r}} \otimes \ldots \otimes \partial_{q_{i_r}} \otimes dq^{j_1} \otimes \ldots \otimes dq^{j_s}, \qquad X|_V = X^i \partial_i.$$

Compute the corresponding expression for the tensor $\mathcal{L}_X t \in \mathcal{T}_s^r(M)$.

Please upload your written solutions in Ilias until Monday, January 18, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.