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Geometry in Physics

Homework Assignment # 8

Problem 29: Lie derivative of tensor fields

Let $t \in \mathcal{T}_s^r(M)$ and $X, Y \in \mathcal{T}_0^1(M)$. Complete the Proof of Proposition 8.10 (in the handwritten lecture notes) which asserts that

$$\mathcal{L}_{[X,Y]}t = \mathcal{L}_X(\mathcal{L}_Y t) - \mathcal{L}_Y(\mathcal{L}_X t).$$

Problem 30: Orthogonal complement

Let V be an n-dimensional real vector space equipped with a pseudometric g and $W \subset V$ a subspace. Prove the following assertions:

(i) $\dim(W) + \dim(W^\perp) = n$ is nondegenerate if and only if $V = W \oplus W^\perp$.

(ii) $(W^\perp)^\perp = W$.

Problem 31: Orthogonal complement of nondegenerate subspaces

Let V be an n-dimensional real vector space equipped with a pseudometric g and $W \subset V$ a subspace. Prove the following assertions:

(i) W is nondegenerate if and only if $V = W \oplus W^\perp$.

(ii) W is nondegenerate if and only if $W^\perp$ is nondegenerate.

(iii) If W is nondegenerate, $\mathrm{ind}(g) = \mathrm{ind}(g|_W) + \mathrm{ind}(g|_{W^\perp})$.

Problem 32: Pullback along a flow

Let $S^{n-1} = \{q \in \mathbb{R}^n \mid \|q\| = 1\} \subset \mathbb{R}^n$. Let $g_{\mathbb{R}^n}$ be the Euclidean metric on $\mathbb{R}^n$ (g_0 in Example 9.10) and $g_{S^{n-1}}$ its restriction to S^{n-1}. Let $I \subset \mathbb{R}$ be an interval, $f \in C^\infty(I)$ be a positive function and

$$M = \left\{(q_0, f(q_0) \cdot q) \in \mathbb{R}^{n+1} \mid q_0 \in I, q \in S^{n-1}\right\} \subset \mathbb{R}^{n+1}.$$

Show that $M \subset \mathbb{R}^{n+1}$ is a smooth submanifold and show that $(M, i^* g_{\mathbb{R}^{n+1}})$ (where $i : M \to \mathbb{R}^{n+1}$ is the inclusion map) is isometric to

$$(J \times S^{n-1}, dt^2 + h(t) g_{S^n}),$$

where $J \subset \mathbb{R}$ is another interval and $h : J \to \mathbb{R}$ another smooth positive function.
Hint: First show that $(M, i^* g_{\mathbb{R}^{n+1}})$ is isometric to $(I \times S^n, f_1(t)dt^2 + f_2(t) g_{S^{n-1}})$ for some positive functions $f_1, f_2 \in C^\infty(I)$.

Please upload your written solutions in Ilias until Monday, January 25, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.