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Geometry in Physics

Homework Assignment # 9

Problem 33: The sphere is conformal to flat space

Consider $S^2 = \{x \in \mathbb{R}^3 \mid \sum(x^i)^2 = 1\} \subset \mathbb{R}^3$, equipped with the metric $g_{S^2} = i^* g_{\mathbb{R}^3}$, where $g_{\mathbb{R}^3}$ is the Euclidean metric and $i : S^2 \to \mathbb{R}^3$ the inclusion map. Let $x = (0,0,1) \in S^2$ be the north pole. Furthermore, consider the map $\phi : \mathbb{R}^2 \to S^2 \setminus \{x\}$, given by

$$(y^1, y^2) \to \left(\frac{2y^1}{1 + (y^1)^2 + (y^2)^2}, \frac{2y^2}{1 + (y^1)^2 + (y^2)^2}, \frac{-1 + (y^1)^2 + (y^2)^2}{1 + (y^1)^2 + (y^2)^2} \right).$$

You may assume that $\phi : \mathbb{R}^2 \to S^2 \setminus \{x\}$ is a diffeomorphism (its inverse is given by the stereographic projection). Show that there is a function $f \in C^\infty(\mathbb{R}^2)$ such that $\phi^* g_{S^2} = f \cdot g_{\mathbb{R}^2}$.

Problem 34: Covariant derivative and isometries

Let (M_1, g_1) and (M_2, g_2) be pseudo-Riemannian manifolds with Levi-Civita connections ${}^{g_1}\nabla$ and ${}^{g_2}\nabla$, respectively. Let $\phi : (M_1, g_1) \to (M_2, g_2)$ be an isometry. Show that

$$\phi_* {}^{g_1}\nabla_X Y = {}^{g_2}\nabla_{\phi_* X} \phi_* Y$$

for all $X, Y \in \mathcal{T}^1_0(M_1)$.

Problem 35: Covariant derivative of conformal metrics

Let (M, g) be a pseudo-Riemannian manifold and $f \in C^\infty(M)$ a positive function. Find a relation between the two covariant derivatives ${}^g\nabla$ and ${}^h\nabla$ of g and $h := f \cdot g$.
Hint: The formula is nicer, if you write $f = e^{2u}$ for some $u \in C^\infty(M)$.

Problem 36: Covariant derivative of tensors

Let (M, g) be a semi-Riemannian manifold and $t \in \mathcal{T}^r_s(M)$ be a tensor field whose coefficients in a chart (V, φ) are given by the functions

$$t^{i_1 \dots i_r}_{j_1 \dots j_s} : V \to \mathbb{R}.$$

Compute the coefficients of $\nabla t \in \mathcal{T}^r_{s+1}(M)$ in the same chart.
Hint: It is reasonable to consider the case of 1-forms first.

Please upload your written solutions in Ilias until Monday, February 01, 12:30 pm. You can submit as a single person or as a group of two people.

For each problem, you get 2,1 or 0 points, depending on the quality (good, medium or poor) of the solution. To be admitted for the exam, you need to get half of the possible points.