Preparatory course for M.Sc. Mathematical Physics (Oct. 2020)

by Stefan Teufel

Contents: 7 video lectures
+ 7 question + exercise sessions

1) Topological, metric, and normed spaces
2) Continuity, compact sets, connected sets
3) Differential calculus
4) Ordinary differential equations
5) Measure and integration theory
6) Classical mechanics
7) Quantum mechanics

Lect. 1: Topological, metric, and normed spaces

Convergence:

A sequence $(x_n) = (x_1, x_2, x_3, \dots)$ in a set X is a map

$$x : \mathbb{N} \to X, \quad n \mapsto x_n$$

<u>Def.:</u> A sequence (x_n) in $\mathbb{R}$ converges to $x \in \mathbb{R}$, iff

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall n \geq N_\varepsilon : \quad |x_n - x| < \varepsilon.$$

(iff for every $\varepsilon > 0$ it holds „eventually" that $|x_n - x| < \varepsilon$)

<u>(x_n) in $\mathbb{R}^n$:</u> — " — : $\|x_n - x\| < \varepsilon$

<u>(x_n) in a metric space:</u> — " — : $d(x_n, x) < \varepsilon$

$:\Leftrightarrow x_n \in B_\varepsilon(x)$

<u>(x_n) in a topol. space:</u> iff for every neighborhood U of x, eventually $x_n \in U$.

<u>Def.:</u> Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

A <u>norm</u> $\|\cdot\|$ on V is a map $\|\cdot\| : V \to [0, \infty)$, $x \mapsto \|x\|$

with the properties:

(i) $\boxed{\|x\| = 0 \iff x = 0}$ ✓

(ii) $\boxed{\forall x \in V, \lambda \in \mathbb{K} : \quad \|\lambda x\| = |\lambda| \cdot \|x\|}$

(iii) $\forall x, y \in V : \quad \|x + y\| \leq \|x\| + \|y\|$

The pair $(V, \|\cdot\|)$ is called a <u>normed space</u>.

<u>Examples:</u> a) On $V = \mathbb{R}^n$ or $\mathbb{C}^n$ the foll. maps are norms:

$$\|x\|_2 := \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2} \qquad \text{(euclidean norm)}$$

$$\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\} \qquad \text{(maximum norm)}$$

$$\|x\|_1 := |x_1| + |x_2| + \dots + |x_n| \qquad \text{(1-norm)}$$

or, more generally, for $p \in [1, \infty)$

$$\|x\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \qquad \text{(p-norm)}$$

b) Let X a set, $(Y, \|\cdot\|_Y)$ a normed space,

and vector space over the same field as Y!

$$V := \{ f : X \to Y \mid \sup_{x \in X} \|f(x)\|_Y < \infty \}.$$

Then
$$\boxed{\|f\|_\infty := \sup_{x \in X} \|f(x)\|_Y}$$

is a norm on V. (Exercise 1: Show this!)

Solution: • $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_Y < \infty$ for all $f \in V$

$\|f\|_\infty \geq 0$ by definition of V.

(i) $\sup_{x \in X} \underbrace{\|f(x)\|_Y}_{\geq 0} = 0$

$$\sup \underbrace{\{ \|f(x)\|_Y \mid x \in X \}}_{\in [0, \infty)}$$

$\iff \|f(x)\|_Y = 0 \quad \forall x \in X$

$\|\cdot\|_Y$ is norm

$\iff f(x) = 0 \quad \forall x \in X$

$\iff: f = 0 \in V$

(ii) Let $f \in V$ and $\lambda \in \mathbb{K}$

$\|\lambda f\|_\infty = \sup \|\lambda f(x)\|$

$A \subset \mathbb{R}$,

A is bounded $:\iff$

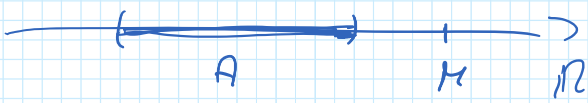
$\exists C \in \mathbb{R} : \forall x \in A \ |x| \leq C$.

M is an upper bound for A

$:\iff \forall x \in A : x \leq M$

$\sup A :=$ least upper bound

Ex: $\sup(0,1) = 1 \notin (0,1)$



Ex: $f : (0,1) \to \mathbb{R}$

$x \mapsto f(x) = x$

(ii) Let $f \in V$ and $\lambda \in \mathbb{K}$

$$\|\lambda f\|_\infty = \sup_{x \in X} \|\lambda f(x)\|_Y$$

$$\overset{\|\cdot\|_Y}{=} \sup_{x\in X} |\lambda| \|f(x)\|_Y$$

$$= |\lambda| \sup_{x \in X} \|f(x)\|_Y$$

$$= |\lambda| \cdot \|f\|_\infty$$

(iii) Let $f, g \in V$, then

$$\|f+g\|_\infty = \sup_{x\in X} \|f(x)+g(x)\|_Y \overset{\|\cdot\|_Y \text{ is norm}}{\leq} \sup_{x \in X} \left(\|f(x)\|_Y + \|g(x)\|_Y \right)$$

$$\leq \sup_{x\in X} \|f(x)\|_Y + \sup_{x \in X} \|g(x)\|_Y = \|f\|_\infty + \|g\|_\infty$$

Ex: $f: (0,1) \to \mathbb{R}$
$x \mapsto f(x) = x$

$\sup_{x \in X} \|f(x)\| = 1$

$\inf A :=$ largest lower bound

M is called the maximum of $A \subset \mathbb{R}$, iff $M \in A$ and $x \leq M$ for all $x \in A$.

Def.: Let X be a set. A metric d on X is a map

$$d: X \times X \to [0, \infty)$$

with the following properties:

(i) $d(x,y) = 0 \iff x = y$

(ii) $\forall x, y \in X: \; d(x,y) = d(y,x)$

(iii) $\forall x, y, z \in X: \; d(x,z) \leq d(x,y) + d(y,z)$

The pair (X, d) is called a metric space.

Examples: a) Let $(V, \|\cdot\|)$ be a normed space. Then

$$d: V \times V \to [0,\infty), \quad (x,y) \mapsto d(x,y) := \|x-y\|$$

defines a metric on V.

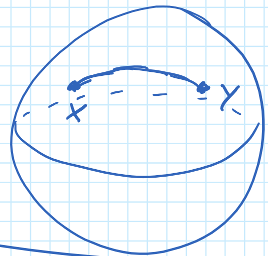
b) Let X be a set. Then the discrete metric on X is

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

c) The euclidean unit sphere $S^2 := \{ x \in \mathbb{R}^3 \mid \|x\|_2 = 1 \}$ with the metric

$$d(x,y) := \arccos(\langle x, y \rangle)$$

is a metric space.



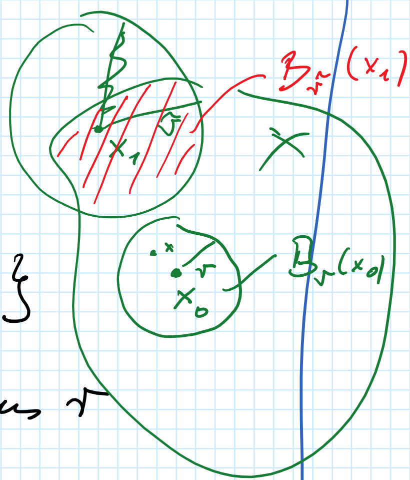
Def.: (Open sets in metric spaces)

Let (X,d) be a metric space.

(a) For $x_0 \in X$ and $r > 0$ the set

$$B_r(x_0) := \{ x \in X \mid d(x,x_0) < r \}$$

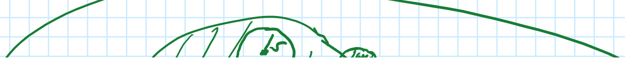
is called the open ball with radius r and center x_0.

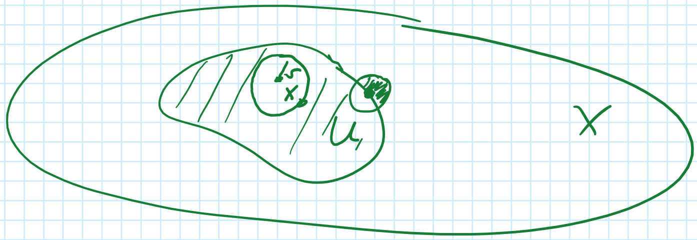


(b) A subset $U \subset X$ is called a neighborhood of $x_0 \in X$, iff U contains an open ball around x_0, i.e. $\exists r > 0 : B_r(x_0) \subset U$.

Then x_0 is called an interior point of U.

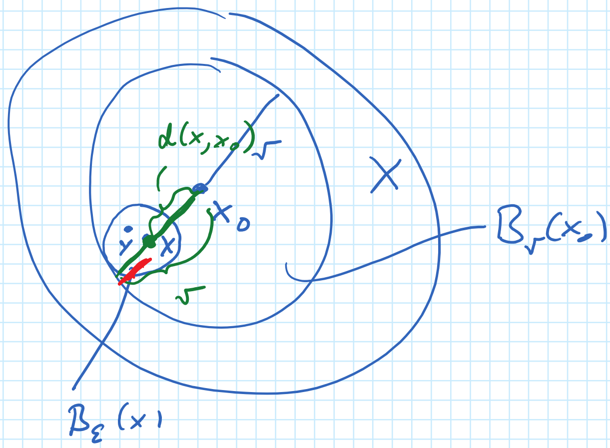
(c) A subset $U \subset X$ is called open, iff it contains only interior points, i.e. iff $\forall x \in U \; \exists r > 0 : B_r(x) \subset U$.





Examples: a) Let (X, d) be a metric space.

Then for any $x_0 \in X$ and $r > 0$ the set $B_r(x_0)$ is open. (Exc. 2: Prove this!)



$\varepsilon := r - d(x, x_0)$

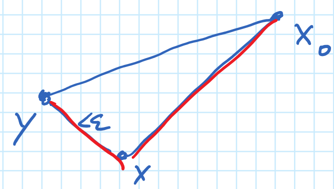
Sol. 2: Let $x \in B_r(x_0)$ and put $\varepsilon := r - \underbrace{d(x, x_0)}_{< r} > 0$.

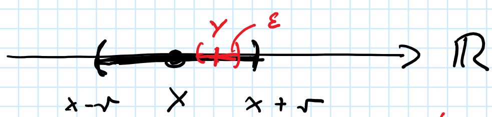
We show that $B_\varepsilon(x) \subset B_r(x_0)$:

Let $y \in B_\varepsilon(x)$ then

$$d(y, x_0) \overset{\Delta\text{-ineq.}}{\leq} d(y, x) + d(x, x_0) < \varepsilon + d(x, x_0) = r.$$

$\Rightarrow y \in B_r(x_0)$.





$\forall y \in B_r(x) \ \exists \varepsilon > 0$
$(y - \varepsilon, y + \varepsilon) \subset (x - r, x + r)$

b) Let X be equipped with the discrete metric.

Then any subset $U \subset X$ is open: $B_{\frac{1}{2}}(x) = \{x\}$ $\forall x \in X$.

Prop.: Let (X,d) be a metric space. Then

(i) $\emptyset$ and X are open.

(ii) If $U, V \subset X$ are open, then also $U \cap V$ is open.

(iii) If $U_i \subset X$ is open for all $i \in I$, then also $\bigcup_{i \in I} U_i$ is open. <u>Ex. 3:</u> Prove this!

<u>Sol. 3:</u> (i) $\emptyset$ is open by definition; X is open because $\forall x \in X$, $\forall r > 0$: $B_r(x) \subset X$.

(ii) Let $U, V \subset X$ be open and let $x \in U \cap V$.
Since U is open, $\exists\, r_1 > 0$: $B_{r_1}(x) \subset U$
Since V -"- , $\exists\, r_2 > 0$: $B_{r_2}(x) \subset V$
Let $r := \min\{r_1, r_2\}$. Then
$B_r(x) \subset \underbrace{B_{r_1}(x)}_{\subset U}$ and $B_r(x) \subset B_{r_2}(x) \subset V$

$\Rightarrow B_r(x) \subset U \cap V$.

(iii) Let $U_i \subset X$, $i \in I$, open and $y \in \bigcup_{i \in I} U_i$.
Then $\exists\, i_0 \in I$: $y \in U_{i_0}$

$\overset{U_{i_0} \text{ open}}{\Longrightarrow}$ $\exists\, r > 0$: $B_r(y) \subset U_{i_0} \subset \bigcup_{i \in I} U_i$

$\Rightarrow \bigcup_{i \in I} U_i$ is open.

(ii) implies that intersections of finitely many open sets are open. However, this is not true for infinitely many sets: Let $U_n := (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$.

Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Def.: A subset $A \subset X$ of a metric space is <u>closed</u>, iff its complement $A^c := \{x \in X \mid x \notin A\}$ is open.

Examples: Let $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$.

Then $[a,b]$, $[a, \infty)$ are closed, but $[a,b)$ is neither open nor closed.

For any metric space (X,d) the sets $\emptyset$ and X are open and closed.

Def.: Let X be a set. A <u>topology</u> $\mathcal{T}$ on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X with the following properties:

(i) $\emptyset, X \in \mathcal{T}$

(ii) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

(iii) $U_i \in \mathcal{T}$ for all $i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$

The sets $U \in \mathcal{T}$ are called the <u>open sets</u> and

The sets $\mathcal{U} \in \mathcal{T}$ are called the open sets and $(X, \mathcal{T})$ is called a topological space.

$A \subset X$ is closed, iff $A^c \in \mathcal{T}$.

For $\mathcal{U} \subset X$ is called a neighbourhood of $x_0 \in X$ (and x_0 an interior point of $\mathcal{U}$), iff $\exists\, \mathcal{O} \in \mathcal{T}$ with $x_0 \in \mathcal{O} \subset \mathcal{U}$.

Example: According to the above Prop., the open sets in a metric space form a topology.

Def: Let $(X, \mathcal{T})$ be a top. space and $Y \subset X$.

(a) The set $\mathring{Y} := \bigcup_{\substack{\mathcal{U} \in \mathcal{T} \\ \mathcal{U} \subset Y}} \mathcal{U}$ is called the interior of Y

(b) The set $\overline{Y} := \bigcap_{\substack{\mathcal{U} \in \mathcal{T} \\ \mathcal{U} \subset Y^c}} \mathcal{U}^c$ is called the closure of Y

(c) The set $\partial Y := \overline{Y} \setminus \mathring{Y}$ is called the boundary of Y.

By definition we have

- $\mathring{Y} \subset Y \subset \overline{Y}$
- $\mathring{Y}$ is the largest open set contained in Y.
- Y is open $\Leftrightarrow Y = \mathring{Y}$
- $\overline{Y}$ is the smallest closed set containing Y.

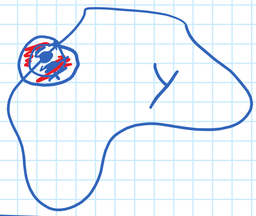
- Y is closed $\iff Y = \overline{Y}$
- $(\mathring{Y})^c = \overline{(Y^c)}$ and $(Y^c)^\circ = (\overline{Y})^c$ (de Morgan)

Prop.: Let $(X, \mathcal{T})$ be a top. space and $Y \subset X$. Then

(a) $\mathring{Y}$ is the set of interior points of Y.

(b) $\boxed{x \in \partial Y \iff \text{for any neighborhood } U \text{ of } x \quad U \cap Y \neq \emptyset \text{ and } U \cap Y^c \neq \emptyset.}$

(c) $\mathring{Y} = Y \setminus \partial Y$ and $\overline{Y} = Y \cup \partial Y$



Example: For $Y = [a, b) \subset \mathbb{R}$ we have

$$\mathring{Y} = (a, b), \quad \overline{Y} = [a, b], \quad \partial Y = \{a, b\}.$$

For $\mathbb{Q} \subset \mathbb{R}$ we have $\mathring{\mathbb{Q}} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial \mathbb{Q} = \mathbb{R}$.

Def.: Let X be a top. space. A sequence (x_n) in X converges to $a \in X$ and we write

$$\lim_{n \to \infty} x_n = a,$$

(X, d)

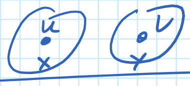
$\iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N$ $x_n \in B_\varepsilon(a)$ ✓

„$B_\varepsilon(a)$“

iff for any neighborhood U of a there exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Note that in general convergence points are not unique: on any set X with the indiscrete topology $\mathcal{T} := \{\emptyset, X\}$ any sequence (x_n) in X converges to every point in X!

Def.: A topological space $(X, \mathcal{T})$ is <u>Hausdorff</u>, iff $\forall x, y \in X,\ x \neq y,\ \exists U, V \in \mathcal{T}: x \in U, y \in V, U \cap V = \emptyset$.



In Hausdorff spaces sequences have at most one limit.

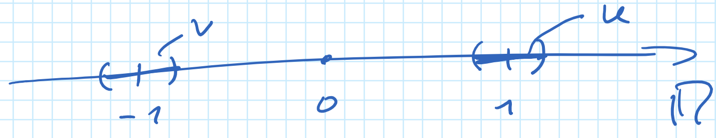
Metric spaces are Hausdorff: for $x, y \in X$, $x \neq y$, we have $d(x,y) =: r > 0$ and $B_{\frac{r}{2}}(x) \cap B_{\frac{r}{2}}(y) = \emptyset$.

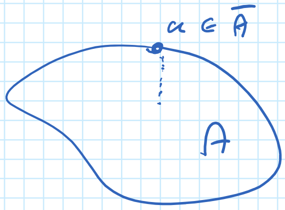
Def.: A point $a \in X$ is called a <u>cluster point</u> of a sequence (x_n) if any neighbourhood U of a contains infinitely many elements of (x_n).

<u>Ex:</u> (x_n) in $\mathbb{R}$: $x_n = (-1)^n$

$$x = (x_1, x_2, x_3, \dots) = (-1, 1, -1, 1, -1, \dots)$$

(x_n) does not converge, but has two cluster points, $\{-1, 1\}$.





Prop.: Let X be a metric space and $A \subset X$. Then

$$a \in \overline{A} \iff \exists\, (x_n) \text{ in } A : \lim_{n\to\infty} x_n = a$$

Note that $\Leftarrow$ holds also in top. spaces.

Def.: A sequence (x_n) in a metric space X is called a <u>Cauchy sequence</u>, iff

uniform spaces

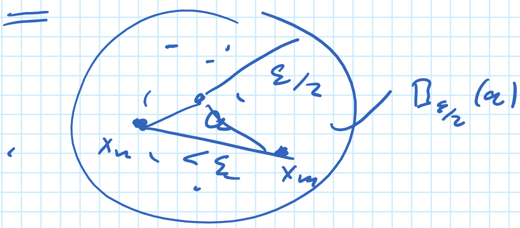
$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N : d(x_n, x_m) < \varepsilon$$

Prop.: Every convergent sequence in a metric space is also Cauchy. Ex. 4: Prove this!

Sol. 4: Let (x_n) in (X, d) and $\lim_{n\to\infty} x_n = a$ and $\varepsilon > 0$.

$$d(x_n, x_m) \leq \underbrace{d(x_n, a)}_{< \frac{\varepsilon}{2}\ \forall n \geq N_{\varepsilon/2}} + \underbrace{d(a, x_m)}_{< \frac{\varepsilon}{2}\ \forall m \geq N_{\varepsilon/2}} < \varepsilon \quad \forall n, m \geq N_{\varepsilon/2}.$$

$\Rightarrow (x_n)$ is Cauchy



Def.: A metric space X is called <u>complete</u>, iff every Cauchy sequence in X converges

iff every Cauchy sequence in X converges.
A complete normed space is called a Banach space.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$ are Banach spaces.

$\to$ $\mathbb{Q}$ is not complete.

(x_n) in $\mathbb{Q}$ with $\lim_{n\to\infty} x_n = \sqrt{2} \notin \mathbb{Q}$
is Cauchy but not convergent in $\mathbb{R}$.

- $(X, \mathcal{T})$ a topological space and $Y \subset X$ a subset (nonempty).
Then $\mathcal{T}|_Y := \{ O \cap Y \mid O \in \mathcal{T} \}$ is a topology on Y, the subspace or relative topology. The elements $U \in \mathcal{T}|_Y$ are called relative open sets.

Examples: • $X = \mathbb{R}$, $Y = [0, 1]$. Then $Y \in \mathcal{T}|_Y$, i.e. Y is relatively open in itself. Also $[0, \frac{1}{2}) \subset Y$ is relat. open, because $[0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap Y$

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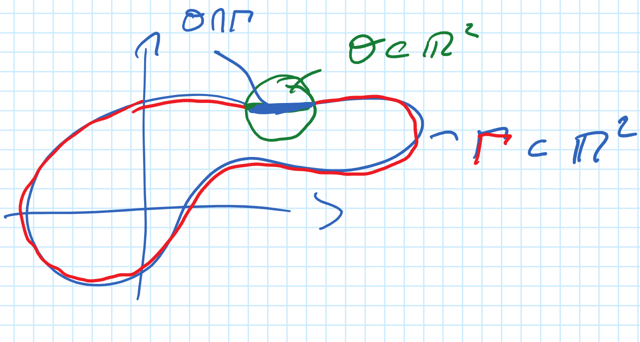

- (X, d) metric space and $Y \subset X$ a subset.
Then $(Y, d|_Y)$ is a metric space

$d|_Y : Y \times Y \to [0,\infty)$, $(y_1, y_2) \mapsto d|_Y(y_1,y_2) := d(y_1,y_2)$

- $(V, \|\cdot\|)$ normed space and $U \subset V$ a vector subspace.

  Then $(U, \|\cdot\||_U)$ is a normed space.